

REGULARIZATION OF p -ADIC STRING AMPLITUDES, AND MULTIVARIATE LOCAL ZETA FUNCTIONS

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ABSTRACT. We prove that the p -adic Koba-Nielsen type string amplitudes are bona fide integrals. We attach to these amplitudes Igusa-type integrals depending on several complex parameters and show that these integrals admit meromorphic continuations as rational functions. Then we use these functions to regularize the Koba-Nielsen amplitudes. As far as we know, there is no a similar result for the Archimedean Koba-Nielsen amplitudes. We also discuss the existence of divergencies and the connections with multivariate Igusa's local zeta functions.

1. INTRODUCTION

This article aims to discuss some connections between p -adic string amplitudes and p -adic local zeta functions (also called Igusa's local zeta functions). In the 80's, Volovich posed the conjecture that the space-time has a non-Archimedean structure at the level of the Planck scale and initiated the p -adic string theory [38], see also [37, Chapter 6], [39]. Volovich noted that the integral expression for the Veneziano amplitude of the open bosonic string can be generalized to a p -adic integral and to an adelic integral giving rise to non-Archimedean Veneziano amplitudes. Then Freund and Witten established (formally) that the ordinary Veneziano and Virasoro-Shapiro four-particle scattering amplitudes can be factored in terms of an infinite product of non-Archimedean string amplitudes [17], see also [3]. As a consequence of the interest on p -adic models of quantum field theory, which is motivated by the fact that these models are exactly solvable, there is a large list of p -adic type Feynman and string amplitudes that are related with local zeta functions of Igusa-type, and it is interesting to mention that seems that the mathematical community working on local zeta functions is not aware of this fact, see e.g. [3], [4], [5], [10], [9], [7], [16], [15], [17], [22], [26], [27], [28], [30], [32], [33], and the references therein.

The connections between Feynman amplitudes and local zeta functions are very old and deep. Let us mention that the works of Speer [34] and Bollini, Giambiagi and González Domínguez [8] on regularization of Feynman amplitudes in quantum field theory are based on the analytic continuation of distributions attached to complex powers of polynomial functions in the sense of Gel'fand and Shilov [18], see also [4], [5], [7], [30], among others. There are several types of local zeta functions, for instance p -adic, Archimedean, topological and motivic, among others, see e.g. [24], [12]-[13] and the references therein. In the Archimedean setting, the local zeta

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functions were introduced in the 50's by Gel'fand and Shilov. The main motivation was that the meromorphic continuation of Archimedean local zeta functions implies the existence of fundamental solutions (i.e. Green functions) for differential operators with constant coefficients. This fact was established, independently, by Atiyah [2] and Bernstein [6]. In the 60's, Weil studied local zeta functions, in the Archimedean and non-Archimedean settings, in connection with the Poisson-Siegel formula [40]. In the 70's, Igusa developed a uniform theory for local zeta functions in characteristic zero [23]-[24]. In the p -adic setting, the local zeta functions are connected with the number of solutions of polynomial congruences mod p^m and with exponential sums mod p^m . Recently Denef and Loeser introduced the motivic zeta functions which constitute a vast generalization of p -adic local zeta functions [13]-[14].

Take $N \geq 4$ and $s_{ij} \in \mathbb{C}$ satisfying $s_{ij} = s_{ji}$ for $1 \leq i < j \leq N-1$. In this article we study the following multivariate Igusa-type zeta function:

$$(1.1) \quad Z^{(N)}(\underline{s}) = \int_{\mathbb{Q}_p^{N-3} \setminus \Lambda} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i,$$

where for $\underline{s} = (s_{ij}) \in \mathbb{C}^D$, $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of \mathbb{Q}_p^{N-3} , and

$$\Lambda := \left\{ (x_2, \dots, x_{N-2}) \in \mathbb{Q}_p^{N-3}; \prod_{i=2}^{N-2} x_i (1 - x_i) \prod_{2 \leq i < j \leq N-2} (x_i - x_j) = 0 \right\}.$$

We call this type of integrals *p -adic open string N -point zeta functions* because they appeared in connection with the p -adic open string N -tachyon tree amplitudes, see e.g. [9], [10], [16], [17], [22], and the references therein. In all the published literature about p -adic string amplitudes, integrals of type (1.1) have been used without considering the convergence of them, i.e. the problem of the regularization of p -adic open string N -tachyon amplitudes has not been considered before. In the light of the theory of local zeta functions, the possible convergence of integrals of type (1.1) is a new and remarkable aspect. The main result of this article, see Theorem 1, establishes that the p -adic open string N -point zeta function is a holomorphic function in a certain domain of \mathbb{C}^D and that it admits an analytic continuation to \mathbb{C}^D (denoted as $Z^{(N)}(\underline{s})$) as a rational function in the variables $p^{-s_{ij}}$, $i, j \in \{1, \dots, N-1\}$. In addition, if $\underline{s} = (s_{ij}) \in \mathbb{R}^D$, with $s_{ij} \geq 0$ for $i, j \in \{1, \dots, N-1\}$, then $Z^{(N)}(\underline{s}) = +\infty$.

At this point, it is worth to mention that the typical approach for establishing that an integral of Igusa-type admits an analytic continuation is via Hironaka's resolution of singularities theorem, see e.g. [24, Chapters 3, 5, 8]. Roughly speaking Hironaka's resolution theorem provides a finite sequence of changes of variables (blow-ups) that allows to express an Igusa-type integral as a linear combination of integrals involving monomials, for this type of integrals the existence of an analytic continuation is easy to show. If the initial Igusa-type integral is a holomorphic function in a certain domain, then by using any suitable sequence of blow-ups the existence of an analytic continuation can be established. If the convergence of the original integral is unknown then, in principle, by using Hironaka's theorem is possible to find an analytic continuation, i.e. a regularization, of the given integral, but this regularization depends on the sequence of blow-ups used, which is not

unique. The problem of showing uniqueness of the regularized integral is highly non-trivial. For this reason, our approach is not based on resolution of singularities, instead of this, we use an approach inspired in the calculations presented in [9] and in the Igusa's p -adic stationary phase formula, see [24, Theorem 10.2.1], [42]-[43]. As a consequence of this approach, all of our results are still valid if we replace \mathbb{Q}_p by $\mathbb{F}_q((t))$, the field of formal Laurent series over a finite field \mathbb{F}_q .

Take $\phi(x_2, \dots, x_{N-2})$ a locally constant function with compact support, then

$$\begin{aligned} & Z_{\phi}^{(N)}(\underline{s}) \\ &= \int_{\mathbb{Q}_p^{N-3} \setminus \Lambda} \phi(x_2, \dots, x_{N-2}) \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i, \end{aligned}$$

is a multivariate Igusa local zeta function. A general theory for this type of local zeta functions was elaborated by Loeser in [29]. In particular, these local zeta functions admit analytic continuations as rational functions of the variables $p^{-s_{ij}}$. If we take ϕ to be the characteristic function of B_r^{N-3} , the ball centered at the origin with radius p^r , the dominated convergence theorem and Theorem 1, imply that $\lim_{r \rightarrow \infty} Z_{B_r^{N-3}}^{(N)}(\underline{s}) = \mathbf{Z}^{(N)}(\underline{s})$ for any \underline{s} in the natural domain of $\mathbf{Z}^{(N)}(\underline{s})$.

In [9], Brekke, Freund, Olson and Witten work out the N -point amplitudes in explicit form and investigate how these can be obtained from an effective Lagrangian. The p -adic open string N -point tree amplitudes are defined as

$$\begin{aligned} (1.2) \quad & \mathbf{A}^{(N)}(\underline{k}) \\ &= \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{\mathbf{k}_1 \mathbf{k}_i} |1 - x_i|_p^{\mathbf{k}_{N-1} \mathbf{k}_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{\mathbf{k}_i \mathbf{k}_j} \prod_{i=2}^{N-2} dx_i, \end{aligned}$$

where $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of \mathbb{Q}_p^{N-3} , $\underline{k} = (\mathbf{k}_1, \dots, \mathbf{k}_N)$, $\mathbf{k}_i = (k_{0,i}, \dots, k_{25,i})$, $i = 1, \dots, N$, $N \geq 4$, is the momentum vector of the i -th tachyon (with Minkowski product $\mathbf{k}_i \mathbf{k}_j = -k_{0,i} k_{0,j} + k_{1,i} k_{1,j} + \dots + k_{25,i} k_{25,j}$) obeying

$$\sum_{i=1}^N \mathbf{k}_i = \mathbf{0}, \quad \mathbf{k}_i \mathbf{k}_i = 2 \text{ for } i = 1, \dots, N.$$

A central problem is to know whether or not integrals of type (1.2) converge for some values $\mathbf{k}_i \mathbf{k}_j \in \mathbb{C}$. Our Theorem 1 allow us to solve this problem. We take the p -adic open string N -point tree integrals $\mathbf{Z}^{(N)}(\underline{s})$ as regularizations of the amplitudes $\mathbf{A}^{(N)}(\underline{k})$. More precisely, we define

$$\mathbf{A}^{(N)}(\underline{k}) = \mathbf{Z}^{(N)}(\underline{s})|_{s_{ij} = \mathbf{k}_i \mathbf{k}_j} \text{ with } i \in \{1, \dots, N-1\}, j \in T \text{ or } i, j \in T,$$

where $T = \{2, \dots, N-2\}$. By Theorem 1, $\mathbf{A}^{(N)}(\underline{k})$ are well-defined rational functions of the variables $p^{-\mathbf{k}_i \mathbf{k}_j}$, $i, j \in \{1, \dots, N-1\}$, which agree with integrals (1.2) when they converge. This definition allows us to recover all the calculations made in [9] and other similar publications. At this point, it is relevant to mention that there is no similar result for the Archimedean string amplitudes at the tree level, as Witten pointed out in [41, p. 4]. We notice that the string amplitudes $\mathbf{A}^{(N)}(\underline{k})$ are limits of local zeta functions when they are considered as distributions, by a

slight abuse of notation, this means that

$$\mathbf{A}^{(N)}(\underline{\mathbf{k}}) = \lim_{r \rightarrow \infty} Z_{B_r^{N-3}}^{(N)}(\underline{\mathbf{k}}),$$

for $\underline{\mathbf{k}}$ in the natural domain of $\mathbf{Z}^{(N)}(\underline{\mathbf{k}})$. Another important problem is to determine the existence of (in the sense of quantum field theory) ultraviolet and infrared divergencies for $\mathbf{A}^{(N)}(\underline{\mathbf{k}})$. If we use the Euclidean product instead of the Minkowski product to define $s_{ij} = \mathbf{k}_i \mathbf{k}_j$, we show that $\mathbf{A}^{(N)}(\underline{\mathbf{k}})$ has infrared divergencies ($\mathbf{A}^{(N)}(0) = +\infty$) and ultraviolet divergencies ($\mathbf{A}^{(N)}(\underline{\mathbf{k}}) = +\infty$ for $\mathbf{k}_i \mathbf{k}_j > 0$). The determination of the ultraviolet and infrared divergencies, in the sense of quantum field theory, in the signature $- + + \dots +$ for $\mathbf{A}^{(N)}(\underline{\mathbf{k}})$ is an open problem. This problem requires the determination of the geometry of the natural domain of function $\mathbf{Z}^{(N)}(\underline{\mathbf{g}})$. This type of problems has been not studied in the case of multivariate local zeta functions.

Lerner and Missarov studied a class of p -adic integrals that includes certain type of Feynman integrals and Koba-Nielsen amplitudes. They showed, see [26, Theorem 2], that this type of integrals can be computed recursively by using hierarchies, but they did not investigate the convergence, or more generally the holomorphy, of the Koba-Nielsen amplitudes, which is a delicate matter. On the other hand, the problem of regularization string amplitudes has been recently considered by Witten in [41], by using an analog of ‘the $i\varepsilon$ method’ for regularizing Feynman integrals. Our approach to the regularization of p -adic string amplitudes is close to the technique of analytic regularization in quantum field theory, see e.g. [34], [25, Chapter 8] and references therein.

Finally, in a more general framework, we point out that the string amplitudes at the tree level and in general the Feynman amplitudes are ‘essentially’ local zeta functions (in the sense of Gel’fand, Sato, Weil, Bernstein, Tate, Igusa, Denef and Loeser among others), and thus, they are algebraic-geometric objects that can be studied over several ground fields, for instance \mathbb{R} , \mathbb{C} , \mathbb{Q}_p , $\mathbb{C}((t))$, and on each of these fields, these objects have similar mathematical properties. As a consequence of our results and the theory of motivic Igusa zeta functions due to Denef and Loeser [13]–[14], it is natural to conjecture the existence of motivic string amplitudes (motivic in the sense of motivic integration) which specialized to the p -adic string amplitudes. In this setting, the limit as p approaches to one of the p -adic string amplitudes must produce topological string amplitudes, which should be string analogues of the topological zeta functions introduced by Denef and Loeser. We recall that “ $\lim_{p \rightarrow 1}$ ” already appeared in several calculations in p -adic string theory, see e.g. [19], [20], thus, we believe that the motivic versions of the string amplitudes may be useful to understand this situation on a solid mathematical ground. In the light of the theory of local zeta functions, it is also natural to conjecture that local zeta functions corresponding to string amplitudes over \mathbb{R} and \mathbb{C} have meromorphic continuations of \mathbb{C}^D . We expect to discuss these matters in forthcoming publications.

The article is organized as follows. In section 2 we present the basic aspects of the p -adic analysis needed in this article, and in section 3, we prove the main result, Theorem 1.

2. ESSENTIAL IDEAS OF p -ADIC ANALYSIS

In this section, we review some ideas and results on p -adic analysis that we will use along this article. For an in-depth exposition, the reader may consult [1], [36], [39].

2.1. The field of p -adic numbers. Along this article p will denote a prime number. The field of p -adic numbers \mathbb{Q}_p is defined as the completion of the field of rational numbers \mathbb{Q} with respect to the p -adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p . The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the p -adic order of x . We extend the p -adic norm to \mathbb{Q}_p^n by taking

$$\|\mathbf{x}\|_p := \max_{1 \leq i \leq n} |x_i|_p, \quad \text{for } \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

We define $\text{ord}(\mathbf{x}) = \min_{1 \leq i \leq n} \{\text{ord}(x_i)\}$, then $\|\mathbf{x}\|_p = p^{-\text{ord}(\mathbf{x})}$. The metric space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a complete ultrametric space. As a topological space \mathbb{Q}_p is homeomorphic to a Cantor-like subset of the real line, see e.g. [1], [39].

Any p -adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{\text{ord}(x)} \sum_{i=0}^{\infty} x_i p^i,$$

where $x_i \in \{0, 1, 2, \dots, p-1\}$ and $x_0 \neq 0$.

For $r \in \mathbb{Z}$, denote by $B_r^n(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n; \|\mathbf{x} - \mathbf{a}\|_p \leq p^r\}$ the ball of radius p^r with center at $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(\mathbf{0}) := B_r^n$. Note that $B_r^n(\mathbf{a}) = B_r(a_1) \times \dots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, the ring of p -adic integers. In addition, $B_r^n(\mathbf{a}) = \mathbf{a} + (p^{-r}\mathbb{Z}_p)^n$. We also denote by $S_r^n(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n; \|\mathbf{x} - \mathbf{a}\|_p = p^r\}$ the sphere of radius p^r with center at $\mathbf{a} \in \mathbb{Q}_p^n$, and take $S_r^n(\mathbf{0}) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of \mathbb{Z}_p), but $(\mathbb{Z}_p^\times)^n \subsetneq S_0^n$, for $n \geq 2$. The balls and spheres are both open and closed subsets in \mathbb{Q}_p^n . In addition, two balls in \mathbb{Q}_p^n are either disjoint or one is contained in the other.

As a topological space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is totally disconnected, i.e. the only connected subsets of \mathbb{Q}_p^n are the empty set and the points. A subset of \mathbb{Q}_p^n is compact if and only if it is closed and bounded in \mathbb{Q}_p^n , see e.g. [39, Section 1.3], or [1, Section 1.8]. The balls and spheres are compact subsets. Thus $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a locally compact topological space.

Remark 1. There is a natural map, called the reduction mod p and denoted as $\overline{\cdot}$, from \mathbb{Z}_p onto \mathbb{F}_p , the finite field with p elements. More precisely, if $x = \sum_{j=0}^{\infty} x_j p^j \in \mathbb{Z}_p$, then $\overline{x} = \overline{x}_0 \in \mathbb{F}_p = \{\overline{0}, \overline{1}, \dots, \overline{p-1}\}$. If $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_p^n$, then $\overline{\mathbf{a}} = (\overline{a}_1, \dots, \overline{a}_n)$.

2.2. Integration on \mathbb{Q}_p^n . Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, there exists a Borel measure dx , called the Haar measure of $(\mathbb{Q}_p, +)$, unique up to multiplication by a positive constant, such that $\int_U dx > 0$ for every non-empty Borel open set $U \subset \mathbb{Q}_p$, and satisfying $\int_{E+z} dx = \int_E dx$ for every Borel set $E \subset \mathbb{Q}_p$,

see e.g. [21, Chapter XI]. If we normalize this measure by the condition $\int_{\mathbb{Z}_p} dx = 1$, then dx is unique. From now on we denote by dx the normalized Haar measure of $(\mathbb{Q}_p, +)$ and by $d^n \mathbf{x}$ the product measure on $(\mathbb{Q}_p^n, +)$.

A function $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is said to be *locally constant* if for every $\mathbf{x} \in \mathbb{Q}_p^n$ there exists an open compact subset U , $\mathbf{x} \in U$, such that $\varphi(\mathbf{x}) = \varphi(\mathbf{u})$ for all $\mathbf{u} \in U$. Any locally constant function $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ can be expressed as a linear combination of characteristic functions of the form $\varphi(\mathbf{x}) = \sum_{k=1}^{\infty} c_k 1_{U_k}(\mathbf{x})$, where $c_k \in \mathbb{C}$ and $1_{U_k}(\mathbf{x})$ is the characteristic function of U_k , an open compact subset of \mathbb{Q}_p^n , for every k . If φ has compact support, then $\varphi(\mathbf{x}) = \sum_{k=1}^L c_k 1_{U_k}(\mathbf{x})$ and in this case

$$\int_{\mathbb{Q}_p^n} \varphi(\mathbf{x}) d^n \mathbf{x} = c_1 \int_{U_1} d^n \mathbf{x} + \dots + c_L \int_{U_L} d^n \mathbf{x}.$$

A locally constant function with compact support is called a *Bruhat-Schwartz function*. These functions form a \mathbb{C} -vector space denoted as $\mathcal{D}(\mathbb{Q}_p^n)$. By using the Stone-Weierstrass theorem, $\mathcal{D}(\mathbb{Q}_p^n)$ is a dense subspace of $C_0(\mathbb{Q}_p^n)$, the space of continuous functions with compact support, and consequently the functional $\varphi \rightarrow \int_{\mathbb{Q}_p^n} \varphi(\mathbf{x}) d^n \mathbf{x}$, $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ has a unique extension to $C_0(\mathbb{Q}_p^n)$. For integrating more general functions, say locally integrable functions, the following notion of improper integral will be used.

Definition 1. A function $\varphi \in L_{loc}^1$ is said to be *integrable in \mathbb{Q}_p^n* if

$$\lim_{m \rightarrow +\infty} \int_{B_m^n(0)} \varphi(\mathbf{x}) d^n \mathbf{x} = \lim_{m \rightarrow +\infty} \sum_{j=-\infty}^m \int_{S_j^n(0)} \varphi(\mathbf{x}) d^n \mathbf{x}$$

exists. If the limit exists, it is denoted as $\int_{\mathbb{Q}_p^n} \varphi(\mathbf{x}) d^n \mathbf{x}$, and we say that the (improper) integral exists.

2.3. Analytic change of variables. A function $h : U \rightarrow \mathbb{Q}_p$ is said to be *analytic* on an open subset $U \subset \mathbb{Q}_p^n$, if for every $\mathbf{b} \in U$ there exists an open subset $\tilde{U} \subset U$, with $\mathbf{b} \in \tilde{U}$, and a convergent power series $\sum_{\mathbf{i}} a_{\mathbf{i}} (\mathbf{x} - \mathbf{b})^{\mathbf{i}}$ for $\mathbf{x} \in \tilde{U}$, such that $h(\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{N}^n} a_{\mathbf{i}} (\mathbf{x} - \mathbf{b})^{\mathbf{i}}$ for $\mathbf{x} \in \tilde{U}$, with $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \cdots x_n^{i_n}$, $\mathbf{i} = (i_1, \dots, i_n)$. In this case, $\frac{\partial}{\partial x_i} h(\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{N}^n} a_{\mathbf{i}} \frac{\partial}{\partial x_i} (\mathbf{x} - \mathbf{b})^{\mathbf{i}}$ is a convergent power series. Let U, V be open subsets of \mathbb{Q}_p^n . A mapping $\mathbf{h} : U \rightarrow V$, $\mathbf{h} = (h_1, \dots, h_n)$ is called *analytic* if each h_i is analytic.

Let $\varphi : V \rightarrow \mathbb{C}$ be a continuous function with compact support, and let $\mathbf{h} : U \rightarrow V$ be an analytic mapping. Then

$$\int_V \varphi(\mathbf{y}) d^n \mathbf{y} = \int_U \varphi(\mathbf{h}(\mathbf{x})) |Jac(\mathbf{h}(\mathbf{x}))|_p d^n \mathbf{x},$$

where $Jac(\mathbf{h}(\mathbf{z})) := \det \left[\frac{\partial h_i}{\partial x_j}(\mathbf{z}) \right]_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$, see e.g. [11, Section 10.1.2].

2.4. The multivariate Igusa zeta functions. Let $f_i(\mathbf{x}) \in \mathbb{Q}_p[x_1, \dots, x_n]$ be a non-constant polynomial for $i = 1, \dots, l$, and let Φ be a Bruhat-Schwartz function. The multivariate local zeta function attached to (f_1, \dots, f_l, Φ) (also called Igusa

local zeta function) is defined by the integral

$$\mathbf{Z}_\Phi(s_1, \dots, s_l; f_1, \dots, f_l) = \int_{\mathbb{Q}_p^n \setminus \bigcup_{i=1}^l f_i^{-1}(\mathbf{0})} \Phi(\mathbf{x}) \prod_{i=1}^l |f_i(\mathbf{x})|_p^{s_i} d^n \mathbf{x}$$

for $(s_1, \dots, s_l) \in \mathbb{C}^n$ with $\operatorname{Re}(s_i) > 0$, $i = 1, \dots, l$. This integral defines a holomorphic function of (s_1, \dots, s_l) in the half-space $\operatorname{Re}(s_i) > 0$, $i = 1, \dots, l$. In the case $l = 1$, this assertion corresponds to Lemma 5.3.1 in [24]. For the general case, we recall that a continuous complex-valued function defined in an open set $A \subseteq \mathbb{C}^n$, which is holomorphic in each variable separately, is holomorphic in A . The multivariate local zeta functions admit analytic continuations to the whole \mathbb{C}^n as rational functions of the variables p^{-s_i} , $i = 1, \dots, l$, see [29]. The Igusa local zeta functions are related with the number of solutions of polynomial congruences mod p^m and with exponential sums mod p^m , there are many intriguing conjectures relating the poles of local zeta functions with the topology of complex singularities, see e.g. [12], [24].

We want to highlight that the convergence of the local zeta functions depends crucially on the fact that Φ has compact support. Consider the following integral:

$$\mathbf{J}(s) = \int_{\mathbb{Q}_p} |x|_p^s dx, \quad s \in \mathbb{C}.$$

Assume that $\mathbf{J}(s_0)$ exists for some $s_0 \in \mathbb{R}$, then necessarily the integrals

$$\mathbf{J}_0(s_0) = \int_{\mathbb{Z}_p} |x|_p^{s_0} dx \quad \text{and} \quad \mathbf{J}_1(s_0) = \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |x|_p^{s_0} dx$$

exist. The first integral is well-known, $\mathbf{J}_0(s_0) = \frac{1-p^{-1}}{1-p^{-1-s_0}}$ for $s_0 > -1$. For the second integral, we use that $|x|_p^{s_0}$ is locally integrable, and thus

$$\mathbf{J}_1(s_0) = \sum_{j=1}^{\infty} \int_{p^{-j}\mathbb{Z}_p^\times} |x|_p^{s_0} dx = \sum_{j=1}^{\infty} p^{j+j s_0} \int_{\mathbb{Z}_p^\times} dx = (1-p^{-1}) \sum_{j=1}^{\infty} p^{j(1+s_0)} < \infty$$

if and only if $s_0 < -1$. Then, integral $\mathbf{J}(s)$ does not exist for any $s \in \mathbb{R}$ and consequently $\mathbf{J}(s)$ does not exist for any complex value s .

For an in-depth discussion on local zeta functions the reader may consult [12], [24] and the references therein.

3. p -ADIC STRING ZETA FUNCTIONS

We fix an integer $N \geq 4$. To each pair (i, j) with $i, j \in \{1, \dots, N-1\}$ we attach a complex number $s_{(i,j)}$ such that $s_{(i,j)} = s_{(j,i)}$. To simplify the notation we will use ij , respectively s_{ij} , instead of (i, j) , respectively, instead of $s_{(i,j)}$. We set $T := \{2, \dots, N-2\}$, $D = \frac{(N-3)(N-4)}{2} + 2(N-3)$ and \mathbb{C}^D as

$$\begin{cases} \{s_{ij} \in \mathbb{C}; i \in \{1, N-1\}, j \in T\} & \text{if } N = 4 \\ \{s_{ij} \in \mathbb{C}; i \in \{1, N-1\}, j \in T \text{ or } i, j \in T \text{ with } i < j\} & \text{if } N \geq 5. \end{cases}$$

We set $\underline{s} = (s_{ij}) \in \mathbb{C}^D$, $\mathbf{x} = (x_2, \dots, x_{N-2}) \in \mathbb{Q}_p^{N-3}$, and

$$F(\underline{s}, \mathbf{x}; N) = \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}}.$$

Definition 2. The p -adic open string N -point zeta function is defined as

$$(3.1) \quad Z^{(N)}(\underline{s}) := \int_{\mathbb{Q}_p^{N-3} \setminus \Lambda} F(\underline{s}, \mathbf{x}; N) \prod_{i=2}^{N-2} dx_i$$

for $\underline{s} = (s_{ij}) \in \mathbb{C}^D$, where

$$\Lambda := \left\{ (x_2, \dots, x_{N-2}) \in \mathbb{Q}_p^{N-3}; \prod_{i=2}^{N-2} x_i (1 - x_i) \prod_{2 \leq i < j \leq N-2} (x_i - x_j) = 0 \right\}$$

and $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of \mathbb{Q}_p^{N-3} .

Remark 2. We notice that the domain of integration in (3.1) is taken to be $\mathbb{Q}_p^{N-3} \setminus \Lambda$ in order to use $a^s = e^{s \ln a}$, with $a > 0$ and $s \in \mathbb{C}$, as the definition of the complex power function. The convergence of integral (3.1), as well as its holomorphy, will be discussed later on.

We define for $I \subseteq T$, the sector attached to I as

$$\text{Sect}(I) = \left\{ (x_2, \dots, x_{N-2}) \in \mathbb{Q}_p^{N-3}; |x_i|_p \leq 1 \Leftrightarrow i \in I \right\}$$

and

$$Z^{(N)}(\underline{s}; I) = \int_{\text{Sect}(I)} F(\underline{s}, \mathbf{x}; N) \prod_{i=2}^{N-2} dx_i.$$

Hence

$$Z^{(N)}(\underline{s}) = \sum_{I \subseteq T} Z^{(N)}(\underline{s}; I).$$

Notation 1. (i) The cardinality of a finite set A will be denoted as $|A|$. (ii) We will use the symbol \sqcup to denote the union of disjoint sets. (iii) Given a non-empty subset I of $\{2, \dots, N-2\}$ and B a non-empty subset of \mathbb{Q}_p , we set

$$B^{|I|} = \{(x_i)_{i \in I}; x_i \in B\}.$$

(iv) By convention, we define $\prod_{i \in \emptyset} \cdot := 1$, $\sum_{i \in \emptyset} \cdot := 0$, and if $J = \emptyset$, then $\int_{B^{|J|}} \cdot := 1$. (v) The indices i, j will run over subsets of T , if we do not specify any subset, we will assume that is T .

Lemma 1. With the above notation the following formulas hold: (i) $F(\underline{s}, \mathbf{x}; N) |_{\text{Sect}(I)} = F_0(\underline{s}, \mathbf{x}; N) F_1(\underline{s}, \mathbf{x}; N)$, where

$$F_0(\underline{s}, \mathbf{x}; N) := \prod_{i \in I} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}}$$

and

$$F_1(\underline{s}, \mathbf{x}; N) := \prod_{i \in T \setminus I} |x_i|_p^{s_{1i} + s_{(N-1)i} + \sum_{\substack{2 \leq j \leq N-2 \\ j \neq i, j \in I}} s_{ij}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in T \setminus I}} |x_i - x_j|_p^{s_{ij}}.$$

(ii) If $\operatorname{Re}(s_{1i}) + \operatorname{Re}(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \operatorname{Re}(s_{ij}) + 1 < 0$ for $i \in T \setminus I$, and $\operatorname{Re}(s_{ij}) > 0$ for $i, j \in T \setminus I$, then

$$\begin{aligned} & \int_{(\mathbb{Q}_p \setminus \mathbb{Z}_p)^{|T \setminus I|}} F_1(\underline{s}, \mathbf{x}; N) \prod_{i \in T \setminus I} dx_i \\ &= p^{M(\underline{s})} \int_{\mathbb{Z}_p^{|T \setminus I|}} \frac{\prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in T \setminus I}} |y_i - y_j|_p^{s_{ij}}}{\prod_{i \in T \setminus I} |y_i|_p^{2 + s_{1i} + s_{(N-1)i} + \sum_{2 \leq j \leq N-2, j \neq i} s_{ij}}} \prod_{i \in T \setminus I} dy_i, \end{aligned}$$

where $M(\underline{s}) := |T \setminus I|_p + \sum_{i \in T \setminus I} (s_{1i} + s_{(N-1)i}) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus I, j \in T}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in I, j \in T \setminus I}} s_{ij}$.

(iii) If $\operatorname{Re}(s_{1i}) + \operatorname{Re}(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \operatorname{Re}(s_{ij}) + 1 < 0$ for $i \in T \setminus I$, $\operatorname{Re}(s_{ij}) > 0$ for $i, j \in T \setminus I$, $\operatorname{Re}(s_{1i}) > 0$ for $i \in I$ and $\operatorname{Re}(s_{(N-1)i}) > 0$ for $i \in I$, then

$$\begin{aligned} Z^{(N)}(\underline{s}; I) &= p^{M(\underline{s})} \left\{ \int_{\mathbb{Z}_p^{|I|}} F_0(\underline{s}, \mathbf{x}; N) \prod_{i \in I} dx_i \right\} \\ &\times \left\{ \int_{\mathbb{Z}_p^{|T \setminus I|}} \frac{\prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in T \setminus I}} |x_i - x_j|_p^{s_{ij}}}{\prod_{i \in T \setminus I} |x_i|_p^{2 + s_{1i} + s_{(N-1)i} + \sum_{2 \leq j \leq N-2, j \neq i} s_{ij}}} \prod_{i \in T \setminus I} dx_i \right\} \\ &=: p^{M(\underline{s})} Z^{(N)}(\underline{s}; I, 0) Z^{(N)}(\underline{s}; T \setminus I, 1). \end{aligned}$$

Remark 3. Later on we will show that the integrals in the right-hand side in the formulas given in (ii) and (iii) are convergent and holomorphic functions on a certain subset of \mathbb{C}^D for all $I \subseteq T$.

Proof. (i) Notice that $F(\underline{s}, \mathbf{x}; N) \big|_{\operatorname{Sect}(I)}$ equals

$$\begin{aligned} & \prod_{i \in I} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{i \in T \setminus I} |x_i|_p^{s_{1i} + s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}} \times \\ (3.2) \quad & \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in T \setminus I}} |x_i - x_j|_p^{s_{ij}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus I, j \in I}} |x_i|_p^{s_{ij}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i \in I, j \in T \setminus I}} |x_j|_p^{s_{ij}}. \end{aligned}$$

Now, by using that $s_{ij} = s_{ji}$,

$$\begin{aligned} & \prod_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus I, j \in I}} |x_i|_p^{s_{ij}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i \in I, j \in T \setminus I}} |x_j|_p^{s_{ij}} = \prod_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus I, j \in I}} |x_i|_p^{s_{ij}} \prod_{\substack{2 \leq j < i \leq N-2 \\ j \in I, i \in T \setminus I}} |x_i|_p^{s_{ij}} \\ (3.3) \quad &= \prod_{\substack{2 \leq j, i \leq N-2 \\ i \neq j, i \in T \setminus I, j \in I}} |x_i|_p^{s_{ij}} = \prod_{i \in T \setminus I} |x_i|_p^{\sum_{\substack{2 \leq j \leq N-2 \\ j \neq i, j \in I}} s_{ij}}. \end{aligned}$$

The announced formula follows from (3.2)-(3.3).

(ii) For $|T \setminus I| \geq 1$, we set

$$\mathbf{I}(\underline{s}; T \setminus I) := \int_{(\mathbb{Q}_p \setminus \mathbb{Z}_p)^{|T \setminus I|}} F_1(\underline{s}, \mathbf{x}; N) \prod_{i \in T \setminus I} dx_i,$$

and for $l \in \mathbb{N} \setminus \{0\}$,

$$(\mathbb{Q}_p \setminus \mathbb{Z}_p)_{-l}^{|T \setminus I|} := \left\{ (x_i)_{i \in T \setminus I} \in (\mathbb{Q}_p \setminus \mathbb{Z}_p)^{|T \setminus I|}; -l \leq \text{ord}(x_i) \leq -1 \text{ for } i \in T \setminus I \right\},$$

$$(p\mathbb{Z}_p)_l^{|T \setminus I|} := \left\{ (x_i)_{i \in T \setminus I} \in (p\mathbb{Z}_p)^{|T \setminus I|}; 1 \leq \text{ord}(x_i) \leq l \text{ for } i \in T \setminus I \right\},$$

and

$$\mathbf{I}_{-l}(\underline{s}; T \setminus I) := \int_{(\mathbb{Q}_p \setminus \mathbb{Z}_p)_{-l}^{|T \setminus I|}} F_1(\underline{s}, \mathbf{x}; N) \prod_{i \in T \setminus I} dx_i.$$

Notice that $(\mathbb{Q}_p \setminus \mathbb{Z}_p)_{-l}^{|T \setminus I|}$, $(p\mathbb{Z}_p)_l^{|T \setminus I|}$ are compact sets and that

$$\begin{aligned} (\mathbb{Q}_p \setminus \mathbb{Z}_p)_{-l}^{|T \setminus I|} &\rightarrow (p\mathbb{Z}_p)_l^{|T \setminus I|} \\ (x_i)_{i \in T \setminus I} &\rightarrow (\sigma(x_i))_{i \in T \setminus I}, \end{aligned}$$

with $\sigma(x_i) = \frac{1}{y_i}$ is an analytic change of variables satisfying $\prod_{i \in T \setminus I} dx_i = \prod_{i \in T \setminus I} \frac{dy_i}{|y_i|_p^2}$, then by using this change of variables and the fact that

$$\begin{aligned} &\prod_{i \in T \setminus I} |y_i|_p^{s_{1i} + s_{(N-1)i} + \sum_{\substack{2 \leq j \leq N-2 \\ j \neq i, j \in I}} s_{ij}} \\ &= \prod_{i \in T \setminus I} |y_i|_p^{s_{1i} + s_{(N-1)i} + \sum_{\substack{2 \leq j \leq N-2 \\ j \neq i, j \in I}} s_{ij}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus I, j \in T \setminus I}} |y_i|_p^{s_{ij}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus I, j \in T \setminus I}} |y_j|_p^{s_{ij}} \\ &= \prod_{i \in T \setminus I} |y_i|_p^{s_{1i} + s_{(N-1)i} + \sum_{\substack{2 \leq j \leq N-2 \\ j \neq i}} s_{ij}} \prod_{i \in T \setminus I} |y_i|_p^{\sum_{\substack{2 \leq j \leq N-2 \\ j \neq i, j \in T \setminus I}} s_{ij}} \\ &= \prod_{i \in T \setminus I} |y_i|_p^{s_{1i} + s_{(N-1)i} + \sum_{\substack{2 \leq j \leq N-2 \\ j \neq i}} s_{ij}}, \end{aligned}$$

we have

$$(3.4) \quad \mathbf{I}_{-l}(\underline{s}; T \setminus I) = \int_{(p\mathbb{Z}_p)_l^{|T \setminus I|}} \frac{\prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in T \setminus I}} |y_i - y_j|_p^{s_{ij}} \prod_{i \in T \setminus I} dy_i}{\prod_{i \in T \setminus I} |y_i|_p^{s_{1i} + s_{(N-1)i} + \sum_{\substack{2 \leq j \leq N-2 \\ j \neq i}} s_{ij} + 2}}.$$

Then $\lim_{l \rightarrow \infty} \mathbf{I}_{-l}(\underline{s}; T \setminus I) = \mathbf{I}(\underline{s}; T \setminus I)$. Indeed, the formula follows from the dominated convergence theorem, by using that $|y_i - y_j|_p^{\text{Re}(s_{ij})} < 1$ for $y_i, y_j \in p\mathbb{Z}_p$, and the fact that $\int_{p\mathbb{Z}_p} \frac{1}{|y|_p^s} dy$ converges for $\text{Re}(s) < 1$. Finally, the announced formula follows from (3.4) by a change of variables.

(iii) It is a consequence of (i)-(ii). \square

Remark 4. From Lemma 1, we have

$$(3.5) \quad \mathbf{Z}^{(N)}(\underline{s}) = \sum_{I \subseteq T} p^{M(\underline{s})} \mathbf{Z}^{(N)}(\underline{s}; I, 0) \mathbf{Z}^{(N)}(\underline{s}; T \setminus I, 1).$$

By convention $\mathbf{Z}^{(N)}(\underline{\mathbf{g}}; \emptyset, 0) = 1$, $\mathbf{Z}^{(N)}(\underline{\mathbf{g}}; \emptyset, 1) = 1$. A central goal of this article is to show that $\mathbf{Z}^{(N)}(\underline{\mathbf{g}})$ has an analytic continuation to the whole \mathbb{C}^D as a rational function in the variables $p^{-s_{ij}}$. To establish this result, we show that all functions appearing on the right-hand side of formula (3.5) admit analytic continuations to the whole \mathbb{C}^D as rational functions in the variables $p^{-s_{ij}}$, and that each of these functions is holomorphic on certain domain, and that the intersection of all these domains contains an open and connected subset of \mathbb{C}^D , which allows us to use the principle of analytic continuation. We will show that each of the integrals $\mathbf{Z}^{(N)}(\underline{\mathbf{g}}; I, 0)$ and $\mathbf{Z}^{(N)}(\underline{\mathbf{g}}; T \setminus I, 1)$ satisfies several recursive formulas, and that by using them, the problem of finding analytic continuations is reduced to case of certain simple integrals.

3.1. Some p -adic integrals. We compute some p -adic integrals needed for calculating $\mathbf{Z}^{(N)}(\underline{\mathbf{g}}; I, 0)$ and $\mathbf{Z}^{(N)}(\underline{\mathbf{g}}; I, 1)$.

Let J be a subset of T with $|J| \geq 2$. We define

$$(3.6) \quad \mathbf{L}_0^{(N)} \left((s_{ij})_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}}; J \right) := \mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; J) = \int_{(\mathbb{Z}_p^\times)^{|J|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i$$

for $\text{Re}(s_{ij}) > 0$ for any ij , and

$$(3.7) \quad \mathbf{L}_1^{(N)} \left((s_{ij})_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}}; J, K \right) := \mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; J, K) = \int_{\mathbb{Z}_p^{|J|}} \prod_{(i, j) \in K} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i$$

where $K \subseteq T_J := \{(i, j) \in T \times T; 2 \leq i < j \leq N-2, i, j \in J\}$ and $\text{Re}(s_{ij}) > 0$ for any ij . Notice that if $|J| = 1$, then $\mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; J) = 1 - p^{-1}$ and $K = \emptyset$ which implies $\mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; J, K) = 1$. A precise definition of integrals $\mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; J)$ requires to integrate on

$$(\mathbb{Z}_p^\times)^{|J|} \setminus \left\{ x \in (\mathbb{Z}_p^\times)^{|J|}; \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} (x_i - x_j) = 0 \right\}.$$

A similar consideration is required for $\mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; J, K)$. However, for the sake of simplicity we use definitions (3.6)-(3.7). We will use this simplified notation later on for similar integrals. The integrals $\mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; J)$, $\mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; J, K)$ are p -adic multivariate local zeta function, these functions were studied by Loeser in [29]. In particular, it is known that these functions have an analytic continuation to \mathbb{C}^D as rational functions in the variables $p^{-s_{ij}}$ and that they are holomorphic functions on $\text{Re}(s_{ij}) > 0$ for any ij .

Remark 5. Let J be subset of T , with $|J| \geq 2$. Set

$$T_J = \{(i, j) \in T \times T; 2 \leq i < j \leq N-2, i, j \in J\}$$

as before. For $\bar{\mathbf{a}} = (\bar{a}_i)_{i \in J} \in (\mathbb{F}_p^\times)^{|J|} \setminus \overline{\Delta}(J)$, with

$$\overline{\Delta}(J) := \left\{ \bar{\mathbf{a}} \in (\mathbb{F}_p^\times)^{|J|}; \bar{a}_i \neq \bar{a}_j \text{ for } i \neq j, \text{ with } i, j \in J \right\},$$

we set

$$K(\bar{\mathbf{a}}) := \{(i, j) \in T_J; \bar{a}_i = \bar{a}_j\}.$$

Now, we introduce on $(\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Delta}(J)$, the following equivalence relation:

$$\bar{\mathbf{a}} \sim \bar{\mathbf{b}} \Leftrightarrow K(\bar{\mathbf{a}}) = K(\bar{\mathbf{b}}).$$

We denote by $\bar{A}(\bar{\mathbf{a}}) = \{\bar{\mathbf{b}} \in (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Delta}(J); \bar{\mathbf{a}} \sim \bar{\mathbf{b}}\}$, the equivalence class defined by $\bar{\mathbf{a}} \in (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Delta}(J)$. For instance, if $\bar{\mathbf{a}} = \bar{\mathbf{1}} = (\bar{1})_{i \in J}$, then $\bar{A}(\bar{\mathbf{1}}) = \bigsqcup_{\bar{\mathbf{b}} \in \mathbb{F}_p^\times} \{\bar{\mathbf{b}}(\bar{\mathbf{1}})_{i \in J}\}$. By taking a unique representative in each equivalence class, we obtain $\mathcal{R}(J) \subset (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Delta}(J)$ such that

$$(\mathbb{F}_p^\times)^{|J|} = \bigsqcup_{\bar{\mathbf{a}} \in \mathcal{R}(J)} \bar{A}(\bar{\mathbf{a}}) \bigsqcup \bar{\Delta}(J).$$

Given a subset $K \subseteq T_J$ with $K = \{(i_1, j_1), \dots, (i_m, j_m)\}$, we define $K_{list} = \{i_1, j_1, \dots, i_m, j_m\} \subset J$. We will use the notation $K_{list}(\bar{\mathbf{a}})$ to mean $K(\bar{\mathbf{a}})_{list}$, for $\bar{\mathbf{a}} \in (\mathbb{F}_p^\times)^{|J|}$. Notice that $K(\bar{\mathbf{a}}) \subset T_{K_{list}(\bar{\mathbf{a}})}$, $|K_{list}(\bar{\mathbf{a}})| \geq 2$ for any $\bar{\mathbf{a}} \in (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Delta}(J)$ and that $K_{list}(\bar{\mathbf{1}}) = J$.

Lemma 2. If $|J| \geq 2$, then, with the notation of Remark 5, the following formula holds:

$$L_0^{(N)}(\underline{\mathbf{s}}; J) = \sum_{\bar{\mathbf{a}} \in \mathcal{R}(J)} |\bar{A}(\bar{\mathbf{a}})| p^{-|J| - \sum_{(i,j) \in K(\bar{\mathbf{a}})} s_{ij}} L_1^{(N)}(\underline{\mathbf{s}}; K_{list}(\bar{\mathbf{a}}), K(\bar{\mathbf{a}})) + |\bar{\Delta}(J)| p^{-|J|}$$

for $\text{Re}(s_{ij}) > 0$ for all $i, j \in J$.

Proof. For $\bar{\mathbf{a}} \in (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Delta}(J)$, set $A(\bar{\mathbf{a}}) := \{\mathbf{b} + p\mathbf{x}; \mathbf{x} \in \mathbb{Z}_p^n, \bar{\mathbf{b}} \in \bar{A}(\bar{\mathbf{a}})\}$, and for $\bar{\Delta}(J)$, $\Delta(J) := \{\mathbf{a} + p\mathbf{x}; \mathbf{x} \in \mathbb{Z}_p^n, \bar{\mathbf{a}} \in \bar{\Delta}(J)\}$. Now

$$\begin{aligned} L_0^{(N)}(\underline{\mathbf{s}}; J) &= \sum_{\bar{\mathbf{a}} \in (\mathbb{F}_p^\times)^{|J|}} \int_{\mathbf{a} + (p\mathbb{Z}_p)^{|J|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i \\ &= \sum_{\bar{\mathbf{a}} \in \mathcal{R}(J)} \sum_{\bar{\mathbf{b}} \in \bar{A}(\bar{\mathbf{a}})} \int_{\mathbf{b} + (p\mathbb{Z}_p)^{|J|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i + \\ &\quad \sum_{\bar{\mathbf{a}} \in \bar{\Delta}(J)} \int_{\mathbf{a} + (p\mathbb{Z}_p)^{|J|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i \\ &= \sum_{\bar{\mathbf{a}} \in \mathcal{R}(J)} |\bar{A}(\bar{\mathbf{a}})| p^{-|J| - \sum_{(i,j) \in K(\bar{\mathbf{a}})} s_{ij}} \int_{(\mathbb{Z}_p)^{|K_{list}(\bar{\mathbf{a}})|}} \prod_{(i,j) \in K(\bar{\mathbf{a}})} |x_i - x_j|_p^{s_{ij}} \prod_{i \in K_{list}(\bar{\mathbf{a}})} dx_i \\ &\quad + |\bar{\Delta}(J)| p^{-|J|}. \end{aligned}$$

□

Lemma 3. We use all the notation introduced in Remark 5. Given $\bar{\mathbf{a}} = (\bar{a}_i)_{i \in J} \in (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Delta}(J)$ and $(i, j) \in K(\bar{\mathbf{a}})$, we set

$$K((i, j), \bar{\mathbf{a}}) := \left\{ \left(\tilde{i}, \tilde{j} \right) \in K(\bar{\mathbf{a}}); \bar{a}_i = \bar{a}_{\tilde{i}} \right\}$$

and use $K_{list}((i, j), \bar{\mathbf{a}}) := K((i, j), \bar{\mathbf{a}})_{list}$. Then the following assertions hold:

- (i) $K((i, j), \bar{\mathbf{a}}) = T_{K_{list}((i, j), \bar{\mathbf{a}})} = \{(r, s); 2 \leq r < s \leq N - 2, r, s \in K_{list}((i, j), \bar{\mathbf{a}})\};$
(ii) the subsets $K((i, j), \bar{\mathbf{a}})$ form a partition of $K(\bar{\mathbf{a}})$, i.e. there exists a finite set $\mathcal{R}(\bar{\mathbf{a}})$ of elements $(i, j) \in K(\bar{\mathbf{a}})$, such that $K(\bar{\mathbf{a}}) = \bigsqcup_{(i, j) \in \mathcal{R}(\bar{\mathbf{a}})} K((i, j), \bar{\mathbf{a}})$.

Proof. (i) By definition $K((i, j), \bar{\mathbf{a}}) \subseteq T_{K_{list}((i, j), \bar{\mathbf{a}})}$. Conversely, let $(\tilde{i}_m, \tilde{j}_l) \in T_{K_{list}((i, j), \bar{\mathbf{a}})}$, then there exists $\tilde{j}_m \in K_{list}((i, j), \bar{\mathbf{a}})$ such that $(\tilde{i}_m, \tilde{j}_m) \in K((i, j), \bar{\mathbf{a}})$ or $(\tilde{j}_m, \tilde{i}_m) \in K((i, j), \bar{\mathbf{a}})$. In any case, $(\tilde{i}_m, \tilde{j}_m)$ or $(\tilde{j}_m, \tilde{i}_m)$ belongs to $K(\bar{\mathbf{a}})$ and $\bar{a}_i = \bar{a}_{\tilde{i}_m} = \bar{a}_{\tilde{j}_m}$. Similarly, there exists $\tilde{i}_l \in K_{list}((i, j), \bar{\mathbf{a}})$ such that $(\tilde{i}_l, \tilde{j}_l)$ or $(\tilde{j}_l, \tilde{i}_l)$ belongs to $K(\bar{\mathbf{a}})$ and $\bar{a}_i = \bar{a}_{\tilde{i}_l} = \bar{a}_{\tilde{j}_l}$. Therefore $\bar{a}_{\tilde{i}_m} = \bar{a}_{\tilde{j}_l}$ i.e. $(\tilde{i}_m, \tilde{j}_l) \in K(\bar{\mathbf{a}})$, and $(\tilde{i}_m, \tilde{j}_l) \in K((i, j), \bar{\mathbf{a}})$. Hence $K((i, j), \bar{\mathbf{a}}) = T_{K_{list}((i, j), \bar{\mathbf{a}})}$.

(ii) Let $(i_m, j_m) \in K((i, j), \bar{\mathbf{a}}) \cap K((\tilde{i}, \tilde{j}), \bar{\mathbf{a}})$, then $\bar{a}_i = \bar{a}_{i_m} = \bar{a}_{\tilde{i}}$ and $(\tilde{i}, \tilde{j}) \in K((i, j), \bar{\mathbf{a}})$, and consequently $K((\tilde{i}, \tilde{j}), \bar{\mathbf{a}}) \subseteq K((i, j), \bar{\mathbf{a}})$. Similarly, one verifies that $K((i, j), \bar{\mathbf{a}}) \subseteq K((\tilde{i}, \tilde{j}), \bar{\mathbf{a}})$. \square

Remark 6. As a consequence of Lemmas 2- 3, we have

$$L_1^{(N)}(\underline{s}; K_{list}(\bar{\mathbf{a}}), K(\bar{\mathbf{a}})) = \prod_{(i, j) \in \mathcal{R}(\bar{\mathbf{a}})} L_1^{(N)}(\underline{s}; K_{list}((i, j), \bar{\mathbf{a}}), T_{K_{list}((i, j), \bar{\mathbf{a}})}).$$

Example 1. Take $p \geq 3$, $\bar{\mathbf{a}} = (\bar{1}, \bar{2}, \bar{1}, \bar{2}, \bar{2}) \in \mathbb{F}_p^5$, and $J = \{2, 3, 4, 5, 6\}$. Hence

$$T_J = \{(2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)\},$$

and by Lemma 3,

$$K(\bar{\mathbf{a}}) = \{(2, 4), (3, 5), (3, 6), (5, 6)\} = K((2, 4), \bar{\mathbf{a}}) \sqcup K((3, 5), \bar{\mathbf{a}}),$$

where $K((2, 4), \bar{\mathbf{a}}) = \{(2, 4)\}$, $K((3, 5), \bar{\mathbf{a}}) = \{(3, 5), (3, 6), (5, 6)\}$. Thus

$$K_{list}((2, 4), \bar{\mathbf{a}}) = \{2, 4\} \text{ and } K_{list}((3, 5), \bar{\mathbf{a}}) = \{3, 5, 6\}.$$

With this notation, $L_1^{(N)}(\underline{s}; K_{list}(\bar{\mathbf{a}}), K(\bar{\mathbf{a}}))$ equals

$$\begin{aligned} & \int_{\mathbb{Z}_p^5} |x_2 - x_4|_p^{s_{24}} |x_3 - x_5|_p^{s_{35}} |x_3 - x_6|_p^{s_{36}} |x_5 - x_6|_p^{s_{56}} dx_2 dx_3 dx_4 dx_5 dx_6 \\ &= \left\{ \int_{\mathbb{Z}_p^2} |x_2 - x_4|_p^{s_{24}} dx_2 dx_4 \right\} \left\{ \int_{\mathbb{Z}_p^3} |x_3 - x_5|_p^{s_{35}} |x_3 - x_6|_p^{s_{36}} |x_5 - x_6|_p^{s_{56}} dx_3 dx_5 dx_6 \right\} \\ &= L_1^{(N)}(\underline{s}; K_{list}((2, 4), \bar{\mathbf{a}}), T_{K_{list}((2, 4), \bar{\mathbf{a}})}) L_1^{(N)}(\underline{s}; K_{list}((3, 5), \bar{\mathbf{a}}), T_{K_{list}((3, 5), \bar{\mathbf{a}})}). \end{aligned}$$

Lemma 4. Set $F(s_1, s_2, s_3, x, y) := |x|_p^{s_1} |y|_p^{s_2} |x - y|_p^{s_3}$, $s_1, s_2, s_3 \in \mathbb{C}$, and

$$Z(s_1, s_2, s_3) := \int_{\mathbb{Z}_p^2} F(s_1, s_2, s_3, x, y) dx dy \text{ for } \operatorname{Re}(s_i) > 0, i = 1, 2, 3.$$

Then $\mathbf{Z}(s_1, s_2, s_3)$ extends holomorphically to a function on

$$\{(s_1, s_2, s_3) \in \mathbb{C}^3; \operatorname{Re}(s_i) > -1 \text{ for } i = 1, 2, 3 \text{ and } \operatorname{Re}(s_1) + \operatorname{Re}(s_2) + \operatorname{Re}(s_3) > -2\}.$$

In addition,

$$\mathbf{Z}(s_1, s_2, s_3) = \frac{Q(p^{-s_1}, p^{-s_2}, p^{-s_3})}{(1 - p^{-2-s_1-s_2-s_3}) \prod_{i=1}^3 (1 - p^{-1-s_i})},$$

where $Q(p^{-s_1}, p^{-s_2}, p^{-s_3})$ denotes a polynomial with rational coefficients in the variables $p^{-s_1}, p^{-s_2}, p^{-s_3}$.

Remark 7. If $s_1 = s_2 = 0$, then the denominator of $\mathbf{Z}(s_1, s_2, s_3)$ is $1 - p^{-1-s_3}$.

Proof. By using that $\mathbb{Z}_p^2 = (p\mathbb{Z}_p)^2 \sqcup S_0^2$ with $S_0^2 = p\mathbb{Z}_p \times \mathbb{Z}_p^\times \sqcup \mathbb{Z}_p^\times \times p\mathbb{Z}_p \sqcup \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$, and then by changing variables, we get

$$\mathbf{Z}(s_1, s_2, s_3) = \frac{\int_{S_0^2} F(s_1, s_2, s_3, x, y) dx dy}{1 - p^{-2-s_1-s_2-s_3}} =: \frac{\mathbf{Z}_0(s_1, s_2, s_3)}{1 - p^{-2-s_1-s_2-s_3}}.$$

On the other hand,

$$\begin{aligned} \mathbf{Z}_0(s_1, s_2, s_3) &= \int_{p\mathbb{Z}_p \times \mathbb{Z}_p^\times} F(s_1, s_2, s_3, x, y) dx dy \\ &+ \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} F(s_1, s_2, s_3, x, y) dx dy + \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} F(s_1, s_2, s_3, x, y) dx dy \\ &=: \mathbf{Z}_{0,1}(s_1, s_2, s_3) + \mathbf{Z}_{0,2}(s_1, s_2, s_3) + \mathbf{Z}_{0,3}(s_1, s_2, s_3). \end{aligned}$$

First, we compute $\mathbf{Z}_{0,1}(s_1, s_2, s_3)$. By a change of variables, we get

$$\mathbf{Z}_{0,1}(s_1, s_2, s_3) = p^{-1-s_1} (1 - p^{-1}) \int_{\mathbb{Z}_p} |x|_p^{s_1} dx = \frac{(1 - p^{-1})^2 p^{-1-s_1}}{1 - p^{-1-s_1}}$$

for $\operatorname{Re}(s_1) > -1$. By a similar computation we obtain

$$\mathbf{Z}_{0,2}(s_1, s_2, s_3) = \frac{(1 - p^{-1})^2 p^{-1-s_2}}{1 - p^{-1-s_2}} \text{ for } \operatorname{Re}(s_2) > -1.$$

In order to compute

$$\mathbf{Z}_{0,3}(s_1, s_2, s_3) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} |x - y|_p^{s_3} dx dy,$$

we use that $(\mathbb{Z}_p^\times)^2 = \sqcup_{\bar{a}_0, \bar{a}_1 \in \mathbb{F}_p^\times} a_0 + p\mathbb{Z}_p \times a_1 + p\mathbb{Z}_p$, where $\mathbb{F}_p^\times = \{1, 2, \dots, p-1\}$ sets, to get

$$\begin{aligned} \mathbf{Z}_{0,3}(s_1, s_2, s_3) &= \sum_{\bar{a}_0, \bar{a}_1 \in \mathbb{F}_p^\times} \int_{a_0 + p\mathbb{Z}_p \times a_1 + p\mathbb{Z}_p} |x - y|_p^{s_3} dx dy \\ &= p^{-2} \sum_{\substack{\bar{a}_0, \bar{a}_1 \in \mathbb{F}_p^\times \\ \bar{a}_0 \neq \bar{a}_1}} \int_{\mathbb{Z}_p \times \mathbb{Z}_p} |a_0 + px - a_1 - py|_p^{s_3} dx dy + p^{-2-s_3} \sum_{\substack{\bar{a}_0, \bar{a}_1 \in \mathbb{F}_p^\times \\ \bar{a}_0 = \bar{a}_1}} \int_{\mathbb{Z}_p \times \mathbb{Z}_p} |x - y|_p^{s_3} dx dy \\ &= p^{-2} (p-1)(p-2) + p^{-2-s_3} (p-1) \frac{1 - p^{-1}}{1 - p^{-1-s_3}}. \end{aligned}$$

□

Lemma 5. *Let I be a subset of T satisfying $|I| \geq 2$. Then $\mathbf{L}_1^{(N)}(\underline{\mathbf{s}}; I, T_I)$ admits an analytic continuation as a rational function of the form*

$$(3.8) \quad \mathbf{L}_1^{(N)}(\underline{\mathbf{s}}; I, T_I) = \frac{Q_I(\{p^{-s_{ij}}\}_{i,j \in I})}{\prod_{J \in \mathcal{F}(I)} \left(1 - p^{-\left(|J|-1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in J}} s_{ij} \right)} \right)^{e_J} \prod_{ij \in S_I} (1 - p^{-1-s_{ij}})^{e_{ij}}},$$

where $Q_I(\{p^{-s_{ij}}\}_{i,j \in I})$ is a polynomial with rational coefficients in the variables $\{p^{-s_{ij}}\}_{i,j \in I}$, $\mathcal{F}(I)$ is a family of subsets of I , with $I \in \mathcal{F}(I)$, S_I is a non-empty subset of $\{2 \leq i < j \leq N-2, i, j \in I\}$, and the e_J, e_{ij} 's are positive integers.

Proof. By using the partition $\mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|}$, where $\mathbb{Z}_p^{|I|} = \{(x_i)_{i \in I}; x_i \in \mathbb{Z}_p\}$, $(p\mathbb{Z}_p)^{|I|} = \{(x_i)_{i \in I}; x_i \in p\mathbb{Z}_p\}$, and $S_0^{|I|} = \{(x_i)_{i \in I} \in \mathbb{Z}_p^{|I|}; \max_{i \in I} \{|x_i|_p\} = 1\}$. By a change of variables, we get

$$\begin{aligned} \mathbf{L}_1^{(N)}(\underline{\mathbf{s}}; I, T_I) &= \frac{\int_{S_0^{|I|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i}{1 - p^{-|I| - \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} s_{ij}}} \\ &=: \frac{\mathbf{A}_0(\underline{\mathbf{s}}; I)}{1 - p^{-|I| - \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} s_{ij}}}. \end{aligned}$$

For every non-empty subset $J \subseteq I$, we define

$$S_J^{|I|} = \left\{ (x_i)_{i \in I} \in \mathbb{Z}_p^{|I|}; |x_i|_p = 1 \Leftrightarrow i \in J \right\},$$

then $S_0^{|I|} = \sqcup_{J \subseteq I, J \neq \emptyset} S_J^{|I|}$ and $\mathbf{A}_0(\underline{\mathbf{s}}; I) = \sum_{J \subseteq I, J \neq \emptyset} \mathbf{A}_{0,J}(\underline{\mathbf{s}})$ where

$$\mathbf{A}_{0,J}(\underline{\mathbf{s}}) := \int_{S_J^{|I|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i,$$

for this reason

$$(3.9) \quad \mathbf{L}_1^{(N)}(\underline{\mathbf{s}}; I, T_I) = \frac{\mathbf{A}_{0,I}(\underline{\mathbf{s}}) + \sum_{J \subseteq I, J \neq \emptyset} \mathbf{A}_{0,J}(\underline{\mathbf{s}})}{1 - p^{-|I| - \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} s_{ij}}}.$$

On the other hand,

$$(3.10) \quad |x_i - x_j|_p^{s_{ij}}|_{S_J^{|I|}} = \begin{cases} |x_i - x_j|_p^{s_{ij}} & \text{if } i, j \in J \\ |x_i - x_j|_p^{s_{ij}} & \text{if } i, j \in I \setminus J \\ 1 & \text{if } i \in J, j \in I \setminus J \\ 1 & \text{if } j \in J, i \in I \setminus J. \end{cases}$$

Then

$$\mathbf{A}_{0,I}(\underline{\mathbf{g}}) = \mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; I),$$

and if $J \subsetneq I$,

$$(3.11) \quad \mathbf{A}_{0,J}(\underline{\mathbf{g}}) = \left\{ \int_{(p\mathbb{Z}_p)^{|I \setminus J|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I \setminus J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I \setminus J} dx_i \right\} \mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; J) \\ = p^{-|I \setminus J| - \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I \setminus J}} s_{ij}} \mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; I \setminus J, T_{I \setminus J}) \mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; J).$$

Therefore, from (3.9)-(3.11), $\mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; I, T_I)$ equals

$$(3.12) \quad \frac{\mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; I) + \sum_{\substack{J \subsetneq I, J \neq \emptyset}} p^{-|I \setminus J| - \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I \setminus J}} s_{ij}} \mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; I \setminus J, T_{I \setminus J}) \mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; J)}{1 - p^{-|I| - \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} s_{ij}}}.$$

Now, by Lemma 2 and the fact that $\overline{A}(\overline{\mathbf{T}}) = \bigsqcup_{\overline{b} \in \mathbb{F}_p^\times} \{(\overline{b})_{i \in I}\}$, $K_{\text{list}}(\overline{\mathbf{T}}) = I$, see Remark 5,

$$(3.13) \quad \mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; I) = \sum_{\overline{\mathbf{a}} \in \mathcal{R}(I) \setminus \{\overline{\mathbf{T}}\}} |\overline{A}(\overline{\mathbf{a}})| p^{-|I| - \sum_{(i,j) \in K(\overline{\mathbf{a}})} s_{ij}} \mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; K_{\text{list}}(\overline{\mathbf{a}}), K(\overline{\mathbf{a}})) \\ + (p-1)p^{-|I| - \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} s_{ij}} \mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; I, T_I) + |\overline{\Delta}(I)| p^{-|I|},$$

with $|K_{\text{list}}(\overline{\mathbf{a}})| \geq 2$, hence from (3.12)-(3.13),

$$\left(1 - p^{1 - |I| - \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} s_{ij}} \right) \mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; I, T_I) \\ = \sum_{\overline{\mathbf{a}} \in \mathcal{R}(I) \setminus \{\overline{\mathbf{T}}\}} d_{\overline{\mathbf{a}}}(\underline{\mathbf{g}}) \mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; K_{\text{list}}(\overline{\mathbf{a}}), K(\overline{\mathbf{a}})) \\ + \sum_{\substack{J \subsetneq I \\ J \neq \emptyset}} c_J(\underline{\mathbf{g}}) \mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; I \setminus J, T_{I \setminus J}) \mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; J) + |\overline{\Delta}(I)| p^{-|I|}.$$

This formula and Lemmas 2-4 give a recursive algorithm for computing integrals $\mathbf{L}_1^{(N)}(\underline{\mathbf{g}}; I, T_I)$, from which we get (3.8). \square

From Lemmas 2-5, we obtain the following result:

Corollary 1. *If $|I| \geq 2$, then*

$$\mathbf{L}_0^{(N)}(\underline{\mathbf{g}}; I) = \frac{R_I\left(\{p^{-s_{ij}}\}_{i,j \in I}\right)}{\prod_{J \in \mathcal{G}(I)} \left(1 - p^{-\left(|J| - 1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in J}} s_{ij}\right)} \right)^{f_J} \prod_{ij \in G_I} (1 - p^{-1-s_{ij}})^{f_{ij}}},$$

where $R_I \left(\{p^{-s_{ij}}\}_{i,j \in I} \right)$ is a polynomial with rational coefficients in the variables $\{p^{-s_{ij}}\}_{i,j \in I}$, $\mathcal{G}(I)$ is a family of non-empty subsets of I , with $I \in \mathcal{G}(I)$, G_I is a non-empty subset of $\{2 \leq i < j \leq N-2, i, j \in I\}$, and the f_J , f_{ij} 's are positive integers.

Given $I \subseteq T$, with $|I| \geq 2$, and $K \subseteq I$, with $|K| \geq 1$, and $M \subseteq T_I$, with $|M| \geq 1$, we define

$$\mathbf{L}_2^{(N)}(\underline{\mathbf{s}}; I, K, M) = \int_{\mathbb{Z}_p^{|I|}} \prod_{i \in K} |x_i|_p^{s_{ti}} \prod_{(i,j) \in M} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i$$

for $\text{Re}(s_{ij}) > 0$ for any ij . If $|M| = 0$, then

$$\mathbf{L}_2^{(N)}(\underline{\mathbf{s}}; I, K, M) = \int_{\mathbb{Z}_p^{|I|}} \prod_{i \in K} |x_i|_p^{s_{ti}} \prod_{i \in I} dx_i.$$

Lemma 6. *Let $t \in \{1, N-1\}$. Then*

$$\mathbf{L}_2^{(N)}(\underline{\mathbf{s}}; I, K, T_I) = \int_{\mathbb{Z}_p^{|I|}} \prod_{i \in K} |x_i|_p^{s_{ti}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i$$

admits an analytic continuation as a rational function of the form

$$(3.14) \quad \mathbf{L}_2^{(N)}(\underline{\mathbf{s}}; I, K, T_I) = \frac{Q_{I,K} \left(\{p^{-s_{ij}}\}_{i,j \in I}, \{p^{-s_{ti}}\}_{t \in \{1, N-1\}, i \in I} \right)}{R_0(\underline{\mathbf{s}}; I, K) R_1(\underline{\mathbf{s}}; I, K) R_2(\underline{\mathbf{s}}; I, K)},$$

where

$$R_0(\underline{\mathbf{s}}; I, K) = \prod_{J \in \mathcal{G}_1(I)} \left(1 - p^{-\left(|J|-1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij} \right)} \right)^{f_J} \prod_{ij \in S_I} (1 - p^{-1-s_{ij}})^{g_{ij}},$$

$$R_1(\underline{\mathbf{s}}; I, K) = \prod_{i \in U_K} (1 - p^{-1-s_{ti}})^{h_i},$$

$$R_2(\underline{\mathbf{s}}; I, K) = \prod_{(J,R) \in \mathcal{G}_2(I \times I)} \left(1 - p^{-|J| - \sum_{i \in R} s_{ti} - \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij}} \right)^{l_{(J,R)}},$$

where $Q_{I,K} \left(\{p^{-s_{ij}}\}_{i,j \in I}, \{p^{-s_{ti}}\}_{t \in \{1, N-1\}, i \in I} \right)$ denotes a polynomial with rational coefficients in the variables $\{p^{-s_{ij}}\}_{i,j \in I}, \{p^{-s_{ti}}\}_{t \in \{1, N-1\}, i \in I}$, $\mathcal{G}_1(I)$ is a non-empty family of subsets of I , with $I \in \mathcal{G}_1(I)$, $\mathcal{G}_2(I \times I)$ is a non-empty family of subsets $J \times R$ of $I \times I$, with $R \subseteq J$ and $(I, K) \in \mathcal{G}_2(I \times I)$, U_K is a non-empty subset of K , S_I is a non-empty subset of $\{2 \leq i < j \leq N-2, i, j \in I\}$, and the f_J 's, g_{ij} 's, h_i 's, and the $l_{(J,R)}$'s are positive integers.

Remark 8. *The integral $\mathbf{L}_2^{(N)}(\underline{\mathbf{s}}; I, K, M)$ is also a multivariate p -adic local zeta function. If $|I| \geq 2$ and $|K| = 0$, then $\mathbf{L}_2^{(N)}(\underline{\mathbf{s}}; I, K, M) = \mathbf{L}_1^{(N)}(\underline{\mathbf{s}}; I, M)$.*

Proof. We use the partition $\mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|}$ as in the proof of Lemma 5 and a change of variables, to get

$$\begin{aligned} L_2^{(N)}(\underline{s}; I, K, T_I) &= \frac{\int \prod_{i \in K} |x_i|_p^{s_{ti}} \prod_{2 \leq i < j \leq N-2, i, j \in I} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i}{1 - p^{-|I| - \sum_{i \in K} s_{ti} - \sum_{2 \leq i < j \leq N-2, i, j \in I} s_{ij}}} \\ &=: \frac{B_0(\underline{s}; I, K, T_I)}{1 - p^{-|I| - \sum_{i \in K} s_{ti} - \sum_{2 \leq i < j \leq N-2, i, j \in I} s_{ij}}}. \end{aligned}$$

We now use the partition $S_0^{|I|} = \sqcup_{J \subseteq I, J \neq \emptyset} S_J^{|I|}$ to obtain

$$B_0(\underline{s}; I, K, T_I) = \sum_{J \subseteq I, J \neq \emptyset} B_{0,J}(\underline{s}),$$

where

$$B_{0,J}(\underline{s}) := \int \prod_{i \in K} |x_i|_p^{s_{ti}} \prod_{2 \leq i < j \leq N-2, i, j \in I} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i.$$

Consequently

$$L_2^{(N)}(\underline{s}; I, K, T_I) = \frac{B_{0,I}(\underline{s}) + \sum_{J \subseteq I, J \neq \emptyset} B_{0,J}(\underline{s})}{1 - p^{-|I| - \sum_{i \in K} s_{ti} - \sum_{2 \leq i < j \leq N-2, i, j \in I} s_{ij}}}.$$

On the other hand, $|x_i - x_j|_p^{s_{ij}}|_{S_J^{|I|}}$ is given in (3.10) and

$$\prod_{i \in K} |x_i|_p^{s_{ti}}|_{S_J^{|I|}} = \prod_{i \in K} |x_i|_p^{s_{ti}}|_{(p\mathbb{Z}_p)^{|K \setminus J|}}.$$

Then $B_{0,I}(\underline{s}) = L_0^{(N)}(\underline{s}; I)$, and if $J \subsetneq I$, $B_{0,J}(\underline{s})$ equals

$$\begin{aligned} &\left\{ \int_{(p\mathbb{Z}_p)^{|I \setminus J|}} \prod_{i \in K \setminus J} |x_i|_p^{s_{ti}} \prod_{2 \leq i < j \leq N-2, i, j \in I \setminus J} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I \setminus J} dx_i \right\} L_0^{(N)}(\underline{s}; J) = \\ &p^{-|I \setminus J| - \sum_{i \in K \setminus J} s_{ti} - \sum_{2 \leq i < j \leq N-2, i, j \in I \setminus J} s_{ij}} \times \end{aligned}$$

$$\left\{ \int_{\mathbb{Z}_p^{|I \setminus J|}} \prod_{i \in K \setminus J} |x_i|_p^{s_{ti}} \prod_{2 \leq i < j \leq N-2, i, j \in I \setminus J} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I \setminus J} dx_i \right\} L_0^{(N)}(\underline{s}; J) =$$

$$p^{-|I \setminus J| - \sum_{i \in K \setminus J} s_{ti} - \sum_{2 \leq i < j \leq N-2, i, j \in I \setminus J} s_{ij}} L_2^{(N)}(\underline{s}; I \setminus J, K \setminus J, T_{I \setminus J}) L_0^{(N)}(\underline{s}; J).$$

Hence $\left(1 - p^{-|I| - \sum_{i \in K} s_{ti} - \sum_{2 \leq i < j \leq N-2, i, j \in I} s_{ij}}\right) L_2^{(N)}(\underline{s}; I, K, T_I)$ equals

$$\begin{aligned} (3.15) \quad &L_0^{(N)}(\underline{s}; I) + \\ &\sum_{J \subsetneq I, J \neq \emptyset} p^{-|I \setminus J| - \sum_{i \in K \setminus J} s_{ti} - \sum_{2 \leq i < j \leq N-2, i, j \in I \setminus J} s_{ij}} \times \\ &L_2^{(N)}(\underline{s}; I \setminus J, K \setminus J, T_{I \setminus J}) L_0^{(N)}(\underline{s}; J). \end{aligned}$$

By using that $|I \setminus J| < |I|$ if $J \subsetneq I$, $J \neq \emptyset$, and that integrals $\mathbf{L}_0^{(N)}(\underline{s}; I)$, $\mathbf{L}_0^{(N)}(\underline{s}; J)$ can be computed effectively, see Corollary 1, formula (3.15) gives a recursive algorithm for computing $\mathbf{L}_2^{(N)}(\underline{s}; I, K, T_I)$, by using it, we obtain (3.14). Notice the integrals of type $\mathbf{L}_2^{(N)}(\underline{s}; I, K, T_I)$, with $|I| = 1$ and $K = \{i\}$ contribute with terms of the form $\frac{1-p^{-1}}{1-p^{-1-s_{ii}}}$. \square

Lemma 7. *Given J a non-empty subset of T , with $|J| \geq 2$, we define*

$$\mathbf{M}_J(\underline{s}; 1) = \int_{(\mathbb{Z}_p^\times)^{|J|}} \prod_{i \in J} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i$$

for $\operatorname{Re}(s_{(N-1)i}) > 0$, $i \in J$, and $\operatorname{Re}(s_{ij}) > 0$, for $i, j \in J$. Then, $\mathbf{M}_J(\underline{s}; 1)$ admits an analytic continuation as a rational function of the form

$$(3.16) \quad \mathbf{M}_J(\underline{s}; 1) = \frac{Q_J \left(\{p^{-s_{ij}}\}_{i, j \in J}, \{p^{-s_{(N-1)i}}\}_{i \in J} \right)}{\prod_{i=1}^3 U_i(\underline{s}; J)},$$

where

$$U_1(\underline{s}; J) = \prod_{M \in \mathcal{F}_1(J)} \left(1 - p^{-\left(|M| - 1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in M}} s_{ij} \right)} \right)^{e_M} \prod_{ij \in S_J^{(1)}} (1 - p^{-1-s_{ij}})^{f_{ij}},$$

$$U_2(\underline{s}; J) = \prod_{(M, S) \in \mathcal{F}_2(J)} \left(1 - p^{-|M| - \sum_{i \in S} s_{(N-1)i} - \sum_{2 \leq i < j \leq N-2, i, j \in M} s_{ij}} \right)^{g_{(M, S)}},$$

and

$$U_3(\underline{s}; J) = \prod_{i \in S_J^{(2)}} (1 - p^{-1-s_{(N-1)i}})^{h_i},$$

where $\mathcal{F}_1(J)$ is a non-empty family of subsets of J , with $J \in \mathcal{F}_1(J)$, $\mathcal{F}_2(J)$ is a non-empty family of subsets $M \times S \subseteq J \times J$, with $S \subseteq M$, $S_J^{(1)}$ and $S_J^{(2)}$ are non-empty subsets of T , and the e_M 's, f_{ij} 's, $g_{(M, S)}$'s and the h_i 's are positive integers.

Remark 9. If $J = \{i\}$, then $\mathbf{M}_J(\underline{s}; 1) = p^{-1} \left(\frac{(1-p^{-1})p^{-s_{(N-1)i}}}{1-p^{-1-s_{(N-1)i}}} + p - 2 \right)$.

Proof. To compute $\mathbf{M}_J(\underline{s}; 1)$, we proceed as follows. We set

$$T_J = \{(i, j) \in T \times T; 2 \leq i < j \leq N-2, i, j \in J\}$$

as before, and for $\bar{\mathbf{a}} = (\bar{a}_i)_{i \in J} \in (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Pi}(J)$, with

$$\bar{\Pi}(J) := \left\{ \bar{\mathbf{a}} \in (\mathbb{F}_p^\times)^{|J|}; \bar{a}_i \neq \bar{a}_j \text{ if } i \neq j, \text{ for } i, j \in J \text{ and } \bar{a}_s \neq 1 \text{ for any } s \in J \right\},$$

we define

$$K(\bar{\mathbf{a}}) = \{(i, j) \in T_J; \bar{a}_i = \bar{a}_j\}, \quad K^{(1)}(\bar{\mathbf{a}}) = \{(i, j) \in T_J; \bar{a}_i = \bar{a}_j = 1\},$$

and

$$K^{(2)}(\bar{\mathbf{a}}) = \{i \in J; \bar{a}_i = 1 \text{ and } \bar{a}_i \neq \bar{a}_s \text{ for any } s \in J \text{ with } s \neq i\}.$$

Notice that $K^{(1)}(\bar{\mathbf{a}}) \subseteq K(\bar{\mathbf{a}})$ and $K^{(2)}(\bar{\mathbf{a}}) \cap K_{\text{list}}(\bar{\mathbf{a}}) = \emptyset$. Now, we introduce on $(\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Pi}(J)$, the following equivalence relation:

$$\bar{\mathbf{a}} \sim \bar{\mathbf{b}} \Leftrightarrow K(\bar{\mathbf{a}}) = K(\bar{\mathbf{b}}) \text{ and } K^{(1)}(\bar{\mathbf{a}}) = K^{(1)}(\bar{\mathbf{b}}) \text{ and } K^{(2)}(\bar{\mathbf{a}}) = K^{(2)}(\bar{\mathbf{b}}).$$

We denote by $\bar{A}(\bar{\mathbf{a}}) = \{\bar{\mathbf{b}} \in (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Pi}(J); \bar{\mathbf{a}} \sim \bar{\mathbf{b}}\}$, the equivalence class defined by $\bar{\mathbf{a}} \in (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Pi}(J)$. By taking a unique representative in each equivalence class, we obtain $\mathcal{R}(J) \subset (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Pi}(J)$ such that

$$(3.17) \quad (\mathbb{F}_p^\times)^{|J|} = \bigsqcup_{\bar{\mathbf{a}} \in \mathcal{R}(J)} \bar{A}(\bar{\mathbf{a}}) \bigsqcup \bar{\Pi}(J).$$

Given a subset $K \subseteq T_J$ with $K = \{(i_1, j_1), \dots, (i_m, j_m)\}$, we define $K_{\text{list}} = \{i_1, j_1, \dots, i_m, j_m\} \subseteq J$ as before. With this notation, $\mathbf{M}_J(\underline{\mathbf{s}}; 1)$ equals

$$(3.18) \quad \sum_{\bar{\mathbf{a}} \in \mathcal{R}(J)} \sum_{\bar{\mathbf{b}} \in \bar{A}(\bar{\mathbf{a}})} \int_{\mathbf{b} + (p\mathbb{Z}_p)^{|J|}} \prod_{i \in J} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i \\ + \sum_{\bar{\mathbf{b}} \in \bar{\Pi}(J)} \int_{\mathbf{b} + (p\mathbb{Z}_p)^{|J|}} \prod_{i \in J} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i \\ := \mathbf{M}(\underline{\mathbf{s}}; J, 1) + \mathbf{M}(\underline{\mathbf{s}}; J, 2).$$

We now use that for each $\bar{\mathbf{a}} \in (\mathbb{F}_p^\times)^{|J|} \setminus \bar{\Pi}(J)$,

$$T_J = K(\bar{\mathbf{a}}) \bigsqcup \{(i, j) \in T_J; \bar{a}_i \neq \bar{a}_j\}$$

and

$$J = K_{\text{list}}^{(1)}(\bar{\mathbf{a}}) \bigsqcup K^{(2)}(\bar{\mathbf{a}}) \bigsqcup \{i \in J; \bar{a}_i \neq \bar{1}\},$$

to obtain

$$\prod_{i \in J} |1 - x_i|_p^{s_{(N-1)i}} = \prod_{i \in K_{\text{list}}^{(1)}(\bar{\mathbf{a}})} |1 - x_i|_p^{s_{(N-1)i}} \prod_{i \in K^{(2)}(\bar{\mathbf{a}})} |1 - x_i|_p^{s_{(N-1)i}}$$

on $\mathbf{b} + (p\mathbb{Z}_p)^{|J|}$, and

$$\prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} = \prod_{(i, j) \in K(\bar{\mathbf{a}})} |x_i - x_j|_p^{s_{ij}}$$

on $\mathbf{b} + (p\mathbb{Z}_p)^{|J|}$. With $J(\bar{\mathbf{a}}) := K^{(2)}(\bar{\mathbf{a}}) \bigsqcup K_{\text{list}}(\bar{\mathbf{a}})$, we have

$$(3.19) \quad \mathbf{M}(\underline{\mathbf{s}}; J, 1) = \sum_{\bar{\mathbf{a}} \in \mathcal{R}(J)} |\bar{A}(\bar{\mathbf{a}})| p^{-|J| - \sum_{i \in K_{\text{list}}^{(1)}(\bar{\mathbf{a}}) \sqcup K^{(2)}(\bar{\mathbf{a}})} s_{(N-1)i} - \sum_{(i, j) \in K(\bar{\mathbf{a}})} s_{ij}} \times \\ \int_{(\mathbb{Z}_p)^{|J(\bar{\mathbf{a}})|}} \prod_{i \in K_{\text{list}}^{(1)}(\bar{\mathbf{a}}) \sqcup K^{(2)}(\bar{\mathbf{a}})} |x_i|_p^{s_{(N-1)i}} \prod_{(i, j) \in K(\bar{\mathbf{a}})} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J(\bar{\mathbf{a}})} dx_i \\ = \sum_{\bar{\mathbf{a}} \in \mathcal{R}(J)} |\bar{A}(\bar{\mathbf{a}})| p^{-|J| - \sum_{i \in K_{\text{list}}^{(1)}(\bar{\mathbf{a}}) \sqcup K^{(2)}(\bar{\mathbf{a}})} s_{(N-1)i} - \sum_{(i, j) \in K(\bar{\mathbf{a}})} s_{ij}} \times \\ \left\{ \int_{(\mathbb{Z}_p)^{|K^{(2)}(\bar{\mathbf{a}})|}} \prod_{i \in K^{(2)}(\bar{\mathbf{a}})} |x_i|_p^{s_{(N-1)i}} \prod_{i \in K^{(2)}(\bar{\mathbf{a}})} dx_i \right\} \mathbf{L}_2^{(N)} \left(\underline{\mathbf{s}}; K_{\text{list}}(\bar{\mathbf{a}}), K_{\text{list}}^{(1)}(\bar{\mathbf{a}}), K(\bar{\mathbf{a}}) \right).$$

Now, by using the partition of $K(\bar{\mathbf{a}})$ given in Lemma 3, we obtain

$$(3.20) \quad L_2^{(N)}(\underline{\mathbf{s}}; K_{\text{list}}(\bar{\mathbf{a}}), K_{\text{list}}^{(1)}(\bar{\mathbf{a}}), K(\bar{\mathbf{a}})) = L_2^{(N)}(\underline{\mathbf{s}}; K_{\text{list}}^{(1)}(\bar{\mathbf{a}}), K_{\text{list}}^{(1)}(\bar{\mathbf{a}}), T_{K_{\text{list}}^{(1)}(\bar{\mathbf{a}})}) \\ \times \prod_{(i,j) \in \mathcal{R}(\bar{\mathbf{a}}) \setminus K^{(1)}(\bar{\mathbf{a}})} L_1^{(N)}(\underline{\mathbf{s}}; K_{\text{list}}((i,j), \bar{\mathbf{a}}), T_{K_{\text{list}}((i,j), \bar{\mathbf{a}})})$$

with the convention that $L_2^{(N)}(\underline{\mathbf{s}}, \emptyset, \emptyset, \emptyset) := 1$. Finally,

$$(3.21) \quad M(\underline{\mathbf{s}}; J, 2) = \sum_{\bar{\mathbf{b}} \in \bar{\Pi}(J)_{\mathbf{b}+(p\mathbb{Z}_p)^{|J|}}} \int \prod_{i \in J} dx_i = p^{-|J|} |\bar{\Pi}(J)|.$$

Hence, formula (3.16) follows from (3.18)-(3.21) by using Lemma 6. \square

3.2. Computation of $Z^{(N)}(\underline{\mathbf{s}}; I, 1)$.

Proposition 1. *Let I be a non-empty subset of T . Then, the integral*

$$Z^{(N)}(\underline{\mathbf{s}}; I, 1) = \begin{cases} \int_{\mathbb{Z}_p^{|I|}} \frac{\prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}}}{\prod_{i \in I} |x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2 \leq j \leq N-2, j \neq i} s_{ij}}} \prod_{i \in I} dx_i & \text{if } |I| \geq 2 \\ \int_{\mathbb{Z}_p} \frac{1}{|x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2 \leq j \leq N-2, j \neq i} s_{ij}}} dx_i & \text{if } |I| = 1 \end{cases}$$

converges on the set

$$\{(s_{ij}) \in \mathbb{C}^D; \text{Re}(s_{ij}) \geq 0 \text{ for } 2 \leq i < j \leq N-2, i, j \in I\} \cap \\ \left\{ (s_{ij}) \in \mathbb{C}^D; 1 + \text{Re}(s_{1i} + s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \text{Re}(s_{ij}) < 0 \text{ for } i \in I \right\},$$

which is an open and connected subset of \mathbb{C}^D . In addition, $Z_p^{(N)}(\underline{\mathbf{s}}; I, 1)$ admits an analytic continuation to \mathbb{C}^D as a rational function of the form

$$(3.22) \quad Z^{(N)}(\underline{\mathbf{s}}; I, 1) = \frac{Q_{I,1}(\{p^{-s_{ij}}; i, j \in \{1, \dots, N-1\}\})}{S_1(\underline{\mathbf{s}}; I) S_2(\underline{\mathbf{s}}; I) S_3(\underline{\mathbf{s}}; I) S_4(\underline{\mathbf{s}}; I)},$$

where $Q_{I,1}(\{p^{-s_{ij}}; i, j \in \{1, \dots, N-1\}\})$ denotes a polynomial with rational coefficients in the variables $p^{-s_{ij}}$, $i, j \in \{1, \dots, N-1\}$,

$$S_1(\underline{\mathbf{s}}; I) :=$$

$$\prod_{J \in \mathcal{H}_1(I)} \left(1 - p^{|J| + \sum_{i \in J} (s_{1i} + s_{(N-1)i}) + \sum_{2 \leq i < j \leq N-2} s_{ij} + \sum_{2 \leq i < j \leq N-2} s_{ij}} \right)^{f_J},$$

where $\mathcal{H}_1(I)$ is a family of non-empty subsets of I , with $I \in \mathcal{H}_1(I)$,

$$S_2(\underline{\mathbf{s}}; I) := \prod_{\substack{J \subseteq I \\ J \neq \emptyset}} \prod_{K \in \mathcal{H}_2(J)} \left(1 - p^{-\left(|K| - 1 + \sum_{2 \leq i < j \leq N-2} s_{ij} \right)} \right)^{e_K},$$

where $\mathcal{H}_2(J)$ is a family of non-empty subsets of J , with $J \in \mathcal{H}_2(J)$, and the e_K 's, f_J 's are positive integers,

$$S_3(\underline{s}; I) := \prod_{\substack{J \subseteq I \\ J \neq \emptyset}} \prod_{ij \in G_J^{(0)}} (1 - p^{-1-s_{ij}}),$$

where $G_J^{(0)}$ is a non-empty subset $\{2 \leq i < j \leq N-2, i, j \in J\}$,

$$S_4(\underline{s}; I) := \prod_{i \in G_I^{(1)}} \left(1 - p^{1+s_{1i}+s_{(N-1)i}+\sum_{2 \leq j \leq N-2, j \neq i} s_{ij}} \right),$$

where $G_I^{(1)}$ is a non-empty subset $\{2 \leq i < j \leq N-2, i, j \in I\}$.

Proof. By using the partition $\mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|}$ as in the proof of Lemma 5, and a change of variables, we get

$$\begin{aligned} Z^{(N)}(\underline{s}; I, 1) &= \frac{\int_{S_0^{|I|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i}{\prod_{i \in I} |x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2 \leq j \leq N-2, j \neq i} s_{ij}}} \\ &= \frac{C_0(\underline{s})}{1 - p^{|I|+\sum_{i \in I} (s_{1i}+s_{(N-1)i}+\sum_{2 \leq i < j \leq N-2} s_{ij}+\sum_{\substack{2 \leq i < j \leq N-2 \\ i \in I}} s_{ij}})}. \end{aligned}$$

We now use the partition $S_0^{|I|} = \sqcup_{J \subseteq I, J \neq \emptyset} S_J^{|I|}$ to obtain

$$C_0(\underline{s}) = \sum_{J \subseteq I, J \neq \emptyset} C_{0,J}(\underline{s}),$$

where

$$C_{0,J}(\underline{s}) := \int_{S_J^{|I|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i,$$

and consequently,

$$Z^{(N)}(\underline{s}; I, 1) = \frac{C_{0,I}(\underline{s}) + \sum_{J \subsetneq I, J \neq \emptyset} C_{0,J}(\underline{s})}{1 - p^{|I|+\sum_{i \in I} (s_{1i}+s_{(N-1)i}+\sum_{2 \leq i < j \leq N-2} s_{ij}+\sum_{\substack{2 \leq i < j \leq N-2 \\ i \in I}} s_{ij}})}.$$

On the other hand, by using (3.10), we have $C_{0,I}(\underline{s}) = L_0^{(N)}(\underline{s}, I)$, and if $J \subsetneq I$,

$$C_{0,J}(\underline{s}) = \left\{ \int_{(p\mathbb{Z}_p)^{|I \setminus J|}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I \setminus J}} |x_i - x_j|_p^{s_{ij}} \prod_{\substack{2 \leq j \leq N-2 \\ j \neq i}} dx_i \prod_{i \in I \setminus J} dx_i \right\} L_0^{(N)}(\underline{s}, J)$$

$$\begin{aligned}
& = p^{|I \setminus J| + \sum_{i \in I \setminus J} (s_{1i} + s_{(N-1)i}) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in I \setminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus (I \setminus J), j \in I \setminus J}} s_{ij}} \times \\
& \left\{ \int_{\mathbb{Z}_p^{|I \setminus J|}} \frac{\prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I \setminus J}} |x_i - x_j|_p^{s_{ij}}}{\prod_{i \in I \setminus J} |x_i|_p^{2 + s_{1i} + s_{(N-1)i} + \sum_{\substack{2 \leq j \leq N-2 \\ j \neq i}} s_{ij}}} \prod_{i \in I \setminus J} dx_i \right\} L_0^{(N)}(\underline{s}, J) \\
& = p^{|I \setminus J| + \sum_{i \in I \setminus J} (s_{1i} + s_{(N-1)i}) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in I \setminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus (I \setminus J), j \in I \setminus J}} s_{ij}} \times \\
& \mathbf{Z}^{(N)}(\underline{s}; I \setminus J, 1) L_0^{(N)}(\underline{s}, J).
\end{aligned}$$

Therefore

(3.23)

$$\mathbf{Z}^{(N)}(\underline{s}; I, 1) = \frac{L_0^{(N)}(\underline{s}; I) + \sum_{J \subsetneq I, J \neq \emptyset} p^{M(\underline{s}, J)} \mathbf{Z}^{(N)}(\underline{s}; I \setminus J, 1) L_0^{(N)}(\underline{s}; J)}{1 - p^{|I| + \sum_{i \in I} (s_{1i} + s_{(N-1)i}) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in I}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus I, j \in I}} s_{ij}}},$$

where

$$\begin{aligned}
M(\underline{s}, J) & := |I \setminus J| + \sum_{i \in I \setminus J} (s_{1i} + s_{(N-1)i}) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in I \setminus J}} s_{ij} \\
& + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus (I \setminus J), j \in I \setminus J}} s_{ij}.
\end{aligned}$$

Notice that in (3.23), $\mathbf{Z}^{(N)}(\underline{s}; I \setminus J, 1)$ may occur with $|I \setminus J| = 1$, say $I \setminus J = \{i\}$, in this case $\mathbf{Z}^{(N)}(\underline{s}; I, 1)$ becomes

(3.24)

$$\int_{\mathbb{Z}_p} \frac{1}{|x_i|_p^{2 + s_{1i} + s_{(N-1)i} + \sum_{2 \leq j \leq N-2, j \neq i} s_{ij}}} dx_i = \frac{1 - p^{-1}}{1 - p^{1 + s_{1i} + s_{(N-1)i} + \sum_{2 \leq j \leq N-2, j \neq i} s_{ij}}}$$

for $\text{Re}(s_{1i}) + \text{Re}(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \text{Re}(s_{ij}) < -1$.

Finally, formula (3.23) gives a recursive algorithm for computing $\mathbf{Z}^{(N)}(\underline{s}; I, 1)$, since $I \setminus J \subsetneq I \subseteq T$ and $L_0^{(N)}(\underline{s}; I)$, $L_0^{(N)}(\underline{s}; J)$ can be effectively computed, see Corollary 1, by using this algorithm and (3.24), we obtain (3.22). \square

Remark 10. Given positive integers N_i , $i \in I \subseteq T$, v , and complex numbers s_i for $i \in I$, we notice that the function $\frac{1}{1 - p^{-v - \sum_{i \in I} N_i s_i}}$ gives rise to a holomorphic function of the s_i on the half-plane $\sum_{i \in I} N_i \text{Re}(s_i) + v > 0$. As a consequence of Proposition 1 there exist families $\mathfrak{F}_1, \mathfrak{F}_2$ of non-empty subsets of T , and a non-empty subset \mathcal{G} of $\{ij; 2 \leq i < j \leq N-2, i, j \in T\}$, such that all the integrals $\mathbf{Z}^{(N)}(\underline{s}; I, 1)$ for all $I \subseteq T$ are holomorphic functions of \underline{s} on the solution set of the conditions:

$$\begin{aligned}
\text{(C1)} \quad & |J| + \sum_{i \in J} (\text{Re}(s_{1i}) + \text{Re}(s_{(N-1)i})) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J}} \text{Re}(s_{ij}) \\
& + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus J, j \in J}} \text{Re}(s_{ij}) < 0 \text{ for } J \in \mathfrak{F}_1;
\end{aligned}$$

$$(C2) \quad |K| - 1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in K}} \operatorname{Re}(s_{ij}) > 0 \text{ for } K \in \mathfrak{F}_2;$$

$$(C3) \quad 1 + \operatorname{Re}(s_{ij}) > 0 \text{ for } ij \in \mathcal{G} \subseteq \{ij; 2 \leq i < j \leq N-2\}.$$

Notice that the condition

$$1 + \operatorname{Re}(s_{1i}) + \operatorname{Re}(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \operatorname{Re}(s_{ij}) < 0$$

is included in Condition C1 taking $|J| = 1$. This fact follows from the following identities:

$$\begin{aligned} \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus J, j \in J}} s_{ij} &= \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T, j \in J}} s_{ij} - \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij} = \\ \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T \setminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T, j \in J}} s_{ij} &= \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T \setminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus J, j \in J}} s_{ij} = \\ \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T \setminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T \setminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij} &= \sum_{\substack{2 \leq i < j \leq N-2 \\ j \neq i, i \in J, j \in T \setminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij}. \end{aligned}$$

Finally, by taking $J = \{i\}$, the last formula becomes $\sum_{\substack{2 \leq j \leq N-2 \\ j \neq i}} s_{ij}$.

Denote by $D_{I,1}$ the natural domain of definition of $\mathbf{Z}^{(N)}(\underline{s}; I, 1)$, i.e. $D_{I,1}$ is an open and connected subset of \mathbb{C}^D in which $\mathbf{Z}^{(N)}(\underline{s}; I, 1)$ is holomorphic and there no exists a larger domain where this property holds.

Lemma 8. Take I to be a non-empty subset of T and set $H_{I,1}(\mathbb{C})$ to be the solution set in \mathbb{C}^D of the following conditions:

$$(3.25) \quad 1 + \operatorname{Re}(s_{1i}) + \operatorname{Re}(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \operatorname{Re}(s_{ij}) < 0, \text{ for } i \in I.$$

Then $D_{I,1}$ is contained in $H_{I,1}(\mathbb{C})$.

Proof. Denote by $H_{I,1}(\mathbb{R})$ the solution set of (3.25) in \mathbb{R}^D . Set $\operatorname{Re}(D_{I,1}) = \{\operatorname{Re}(s_{ij}) \in \mathbb{R}^D; (s_{ij}) \in D_{I,1}\}$. With this notation, it is sufficient to show that $\operatorname{Re}(D_{I,1}) \subset H_{I,1}(\mathbb{R})$. In order to do this, we show that $\mathbf{Z}^{(N)}(\underline{\tilde{s}}; I, 1)$ diverges to $+\infty$ for any $\underline{\tilde{s}} \in \mathbb{R}^D \setminus H_{I,1}(\mathbb{R})$. We prove this last assertion by contradiction. Assume that $\mathbf{Z}^{(N)}(\underline{\tilde{s}}; I, 1) < +\infty$ for $\underline{\tilde{s}} = (\tilde{s}_{ij}) \in \mathbb{R}^D$ with $\tilde{s}_{ij} \geq 0$ for $2 \leq i < j \leq N-2$, $i, j \in I$ and that $\underline{\tilde{s}} \notin H_{I,1}(\mathbb{R})$. This last condition implies that at least a condition of the form

$$(3.26) \quad 1 + \tilde{s}_{1i_0} + \tilde{s}_{(N-1)i_0} + \sum_{2 \leq j \leq N-2, j \neq i_0} \tilde{s}_{ij} \geq 0$$

for some $i_0 \in I$, holds. Then, from $\mathbf{Z}^{(N)}(\underline{\tilde{s}}; I, 1) < +\infty$, we have

$$I(\underline{\tilde{s}}; A) := \int_A \frac{\prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{\tilde{s}_{ij}}}{\prod_{i \in I} |x_i|_p^{2 + \tilde{s}_{1i} + \tilde{s}_{(N-1)i} + \sum_{2 \leq j \leq N-2, j \neq i} \tilde{s}_{ij}}} \prod_{i \in I} dx_i < +\infty$$

for any measurable subset A of $\mathbb{Z}_p^{|I|}$. Take

$$A_0 = \left\{ (x_i)_{i \in I} \in \mathbb{Z}_p^{|I|}; |x_{i_0}|_p < 1 \text{ and } |x_i|_p = 1 \text{ for } i \in I \setminus \{i_0\} \right\}.$$

Then, by (3.26) and some $\epsilon \geq 0$,

$$I(\tilde{\mathbf{g}}; A_0) = \int_A \frac{\prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I \setminus \{i_0\}}} |x_i - x_j|_p^{\tilde{s}_{ij}}}{|x_{i_0}|_p^{1+\epsilon}} \prod_{i \in I} dx_i = +\infty.$$

Therefore, if $\mathbf{Z}^{(N)}(\tilde{\mathbf{g}}; I, 1) < +\infty$, necessarily $\tilde{\mathbf{g}} \in H_{I,1}(\mathbb{R})$. \square

Corollary 2. *If $\mathbf{s} = (s_{ij}) \in \mathbb{R}^D$, with $s_{ij} \geq 0$ for $i, j \in \{1, \dots, N-1\}$, then $\mathbf{Z}^{(N)}(\mathbf{s}; I, 1) = +\infty$, for any non-empty subset I of T .*

3.3. Computation of $\mathbf{Z}^{(N)}(\mathbf{s}; I, 0)$.

Proposition 2. *Let I be a subset of T satisfying $|I| \geq 2$. Then, the integral*

$$\mathbf{Z}^{(N)}(\mathbf{s}; I, 0) = \int_{\mathbb{Z}_p^{|I|}} \prod_{i \in I} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i$$

gives rise to a holomorphic function on

$$H_{I,0} := \left\{ (s_{ij}) \in \mathbb{C}^D; \operatorname{Re}(s_{ij}) > 0 \text{ for } i, j \in I \right\} \cap \left\{ (s_{ij}) \in \mathbb{C}^D; \operatorname{Re}(s_{1i}) > 0 \text{ for } i \in I \right\} \\ \cap \left\{ (s_{ij}) \in \mathbb{C}^D; \operatorname{Re}(s_{(N-1)i}) > 0 \text{ for } i \in I \right\},$$

which is an open and connected subset of \mathbb{C}^D . Furthermore $\mathbf{Z}_p^{(N)}(\mathbf{s}; I, 0)$ has an analytic continuation as a rational function of the form

$$\mathbf{Z}^{(N)}(\mathbf{s}; I, 0) = \frac{Q_{I,0}(\{p^{-s_{1i}}, p^{-s_{(N-1)i}}, p^{-s_{ij}}; i, j \in T\})}{\prod_{i=0}^2 R_i(\mathbf{s}; I, I) \prod_{i=1}^3 U_i(\mathbf{s}; I)},$$

where $Q_{I,0}(\{p^{-s_{1i}}, p^{-s_{(N-1)i}}, p^{-s_{ij}}; i, j \in T\})$ is a polynomial in the variables $p^{-s_{1i}}, p^{-s_{(N-1)i}}, p^{-s_{ij}}$ for $i, j \in T$, $U_i(\mathbf{s}; I)$, $i = 1, 2, 3$ are as in Lemma 7,

$$R_1(\mathbf{s}; I, I) = \prod_{i \in U_I} (1 - p^{-1-s_{1i}})^{h_i},$$

$$R_2(\mathbf{s}; I, K) = \prod_{(J,R) \in \mathcal{G}_2(I \times I)} \left(1 - p^{-|J| - \sum_{i \in R} s_{1i} - \sum_{2 \leq i < j \leq N-2, i, j \in J} s_{ij}} \right)^{l_{(J,R)}},$$

$R_0(\mathbf{s}; I, I)$, $\mathcal{G}_2(I \times I)$ are as in Lemma 6, and the $l_{(J,R)}$'s are positive integers.

Proof. By using that $\mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|}$, we have

$$(3.27) \quad \mathbf{Z}^{(N)}(\mathbf{s}; I, 0) = \mathbf{M}_1^{(N)}(\mathbf{s}; I) + \mathbf{M}_2^{(N)}(\mathbf{s}; I),$$

where

$$\mathbf{M}_1^{(N)}(\mathbf{s}; I) := \int_{(p\mathbb{Z}_p)^{|I|}} \prod_{i \in I} |x_i|_p^{s_{1i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx, \\ \mathbf{M}_2^{(N)}(\mathbf{s}; I) := \int_{S_0^{|I|}} \prod_{i \in I} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx.$$

Now, by changing variables and using Lemma 6 with $t = 1$, $M_1^{(N)}(\underline{s}; I)$ equals

$$(3.28) \quad p^{-|I| - \sum_{i \in I} s_{1i} - \sum_{2 \leq i < j \leq N-2} s_{ij}} \int_{\mathbb{Z}_p^{|I|}} \prod_{i \in I} |x_i|_p^{s_{1i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i \\ = p^{-|I| - \sum_{i \in I} s_{1i} - \sum_{2 \leq i < j \leq N-2} s_{ij}} L_2^{(N)}(\underline{s}; I, I, T_I).$$

To compute $M_2^{(N)}(\underline{s}; I)$, we use the partition $S_0^{|I|} = \sqcup_{J \subseteq I, J \neq \emptyset} S_J^{|I|}$, with $S_J^{|I|} = \{(x_i)_{i \in I} \in \mathbb{Z}_p^{|I|}; |x_i|_p = 1 \Leftrightarrow i \in J\}$, then $M_2^{(N)}(\underline{s}; I)$ equals

$$(3.29) \quad \sum_{\substack{J \subseteq I \\ J \neq \emptyset}} \int \prod_{i \in I} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i \\ = \sum_{\substack{J \subseteq I \\ J \neq \emptyset}} M_J(\underline{s}),$$

where

$$M_J(\underline{s}) = \int_{S_J^{|I|}} \prod_{i \in I \setminus J} |x_i|_p^{s_{1i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I \setminus J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} |1 - x_i|_p^{s_{(N-1)i}} \times \\ \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i = \int_{(p\mathbb{Z}_p)^{|I \setminus J|}} \prod_{i \in I \setminus J} |x_i|_p^{s_{1i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I \setminus J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I \setminus J} dx_i \\ \times \int_{(\mathbb{Z}_p^\times)^{|J|}} \prod_{i \in J} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i := M_{I \setminus J}(\underline{s}; 0) M_J(\underline{s}; 1).$$

We notice that if $J = I$, then, by convention, $M_{I \setminus J}(\underline{s}; 0) = 1$. Now suppose that $J \subsetneq I$. From Lemma 6 with $t = 1$, we have

$$(3.30) \quad M_{I \setminus J}(\underline{s}; 0) = p^{-|I \setminus J| - \sum_{i \in I \setminus J} s_{1i} - \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in I \setminus J}} s_{ij}} L_2^{(N)}(\underline{s}; I \setminus J, I \setminus J, T_{I \setminus J}).$$

The announced result follows from formulas (3.27)-(3.30), and $M_J(\underline{s}; 1)$ by using Lemmas 6-7 and Remark 9. \square

Remark 11. As a consequence of Proposition 2 all the integrals $Z^{(N)}(\underline{s}; I, 0)$ for all $I \subseteq T$ are holomorphic functions of \underline{s} on the solution set in \mathbb{C}^D of the following conditions:

$$(C4) \quad |J| + \sum_{i \in S} \operatorname{Re}(s_{ti}) + \sum_{2 \leq i < j \leq N-2, i, j \in J} \operatorname{Re}(s_{ij}) > 0 \text{ for } J \times S \in \mathfrak{F}_3,$$

with $S \subseteq J$, $t \in \{1, N-1\}$, and \mathfrak{F}_3 a family of non-empty subsets of $I \times I$;

$$(C5) \quad |K| - 1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in K}} \operatorname{Re}(s_{ij}) > 0 \text{ for } K \in \mathfrak{F}_4,$$

where \mathfrak{F}_4 is a family of non-empty subsets of I ;

$$(C6) \quad 1 + \operatorname{Re}(s_{ij}) > 0 \text{ for } ij \in G_T,$$

where G_T is a non-empty subset of $\{2 \leq i < j \leq N-2, i, j \in J\}$ with $(N-1)i, 1i \in G_T$.

Remark 12. If $\underline{s} = (0)_{ij}$ for $i, j \in \{1, \dots, N-1\}$, then $Z^{(N)}(\underline{0}; I, 0) = 1$, for any non-empty subset I of T .

Definition 3. Denote by $H(\mathbb{R})$, respectively by $H(\mathbb{C})$, the solution set of conditions C1-C6 in \mathbb{R}^D , respectively in \mathbb{C}^D .

3.4. Main Theorem.

Lemma 9. Consider the following conditions:

$$(C'1) \quad |J| + \sum_{i \in J} (\operatorname{Re}(s_{1i}) + \operatorname{Re}(s_{(N-1)i})) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J}} \operatorname{Re}(s_{ij}) \\ + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \setminus J, j \in J}} \operatorname{Re}(s_{ij}) < 0 \text{ for } J \subseteq T, |J| \geq 1;$$

$$(C'2) \quad |J| - 1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} \operatorname{Re}(s_{ij}) > 0 \text{ for } J \subseteq T, |J| \geq 2;$$

$$(C'3) \quad |J| + \sum_{i \in S} \operatorname{Re}(s_{ti}) + \sum_{2 \leq i < j \leq N-2, i, j \in J} \operatorname{Re}(s_{ij}) > 0 \\ \text{for } t \in \{1, N-1\}, J \times S \subseteq T \times T \text{ with } |J| \geq 2 \text{ and } |S| \geq 1, S \subseteq J;$$

$$(C'4) \quad 1 + \operatorname{Re}(s_{ij}) > 0 \text{ for } ij \in \{(i, j); 1 \leq i < j \leq N-1\}.$$

Denote by $H_0(\mathbb{R})$, respectively by $H_0(\mathbb{C})$, the solution set of conditions C'1-C'4 in \mathbb{R}^D , respectively in \mathbb{C}^D . Then $H_0(\mathbb{R})$ is convex and bounded set with non-empty interior, and $H_0(\mathbb{C})$ contains an open and connected subset of \mathbb{C}^D . Furthermore, $H_0(\mathbb{R}) \subset H(\mathbb{R})$ and $H_0(\mathbb{C}) \subset H(\mathbb{C})$.

Proof. We first notice that for all $N \geq 4$, the solution set $H_0(\mathbb{R})$ is an open convex set because it is a finite intersection of open half-spaces.

Claim $H_0(\mathbb{R})$ is a non-empty bounded subset. We consider the case $N \geq 5$ in which $|T| \geq 2$. Set $N_1 = \frac{(N-4)(N-3)}{2}$. We define, for $i, j \in \{2, \dots, N-2\}$, the following conditions:

$$(C''1) \quad -\frac{2}{3N_1} < \operatorname{Re}(s_{ij}) < 0,$$

$$(C''2) \quad -\frac{2}{3} < \operatorname{Re}(s_{1i}) < -\frac{1}{2},$$

$$(C''3) \quad -\frac{2}{3} < \operatorname{Re}(s_{(N-1)i}) < -\frac{1}{2}.$$

We notice that the solution set of conditions C''1-C''3 is a non-empty open and connected subset in \mathbb{R}^D . We now verify that the conditions C''1-C''2 imply conditions C'1-C'4. First, consider $J \subseteq T$ such that $|J| = 1$. We can assume that $J = \{i_0\}$ for some $i_0 \in T$. By conditions C''1-C''3, we have

$$(3.31) \quad 1 + \operatorname{Re}(s_{1i_0}) + \operatorname{Re}(s_{(N-1)i_0}) < 1 - 1/2 - 1/2 = 0,$$

$$(3.32) \quad \sum_{2 \leq i_0 < j \leq N-2} \operatorname{Re}(s_{i_0 j}) + \sum_{\substack{2 \leq i < i_0 \leq N-2, \\ i \in T \setminus J}} \operatorname{Re}(s_{ii_0}) < 0,$$

thus, C'1 follows from (3.31) and (3.32). Conditions C'2, C'3 and C'4 follow directly from C''1–C''3.

We now consider $J \subseteq T$ such that $|J| \geq 2$. Condition C'1 is obtained with a similar calculation to (3.31) and (3.32). Now, by condition C''1, we get

$$|J| - 1 + \sum_{2 \leq i < j \leq N-2, i, j \in J} \operatorname{Re}(s_{ij}) > |J| - 1 - \frac{2}{3} > |J| - \frac{5}{3} > 0,$$

which implies C'2. We now verify Condition C'3. Let $t \in \{1, N-1\}$, by using conditions C''2 and C''3,

$$\begin{aligned} & |J| + \sum_{i \in S} \operatorname{Re}(s_{ti}) + \sum_{2 \leq i < j \leq N-2, i, j \in J} \operatorname{Re}(s_{ij}) \\ & > |J| - \frac{2}{3}|S| - \frac{2|(i, j); 2 \leq i < j \leq N-2, i, j \in J|}{3N_1} \\ & \geq |J| - \frac{2}{3}|S| - \frac{2}{3}. \end{aligned}$$

There are two cases. First, $|S| = 1$. In this case $|J| - \frac{2}{3}|S| - \frac{2}{3} > 0$. If $|S| \geq 2$, by using $-\frac{2}{3}|S| - \frac{2}{3} \geq -|S|$ and $|J| \geq |S|$, then $|J| - \frac{2}{3}|S| - \frac{2}{3} \geq |J| - |S| \geq 0$.

Finally, conditions C'4 follows from conditions C''1–C''3. Therefore, $H_0(\mathbb{R})$ is convex and bounded set with non-empty interior, and $H_0(\mathbb{C})$ contains an open and connected subset of \mathbb{C}^D . Finally, since conditions C'1–C'4 imply conditions C1–C6, we conclude that $H_0(\mathbb{R}) \subset H(\mathbb{R})$ and that $H_0(\mathbb{C}) \subset H(\mathbb{C})$.

In the case $N = 4$, $|T| = 1$, the verification of the claim is straightforward. \square

Theorem 1. (1) The p -adic open string N -point zeta function, $\mathbf{Z}^{(N)}(\underline{s})$, gives rise to a holomorphic function on $H(\mathbb{C})$, which contains a non-empty open and connected subset of \mathbb{C}^D . Furthermore, $\mathbf{Z}^{(N)}(\underline{s})$ admits an analytic continuation to \mathbb{C}^D , denoted also as $\mathbf{Z}^{(N)}(\underline{s})$, as a rational function in the variables $p^{-s_{ij}}$, $i, j \in \{1, \dots, N-1\}$. The real parts of the poles of $\mathbf{Z}^{(N)}(\underline{s})$ belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1–C6 with the symbols ' $<$ ', ' $>$ ' replaced by ' $=$ '. (2) If $\underline{s} = (s_{ij}) \in \mathbb{C}^D$, with $\operatorname{Re}(s_{ij}) \geq 0$ for $i, j \in \{1, \dots, N-1\}$, then $\mathbf{Z}^{(N)}(\underline{s}) = +\infty$.

Proof. (1) We recall that

$$(3.33) \quad \mathbf{Z}^{(N)}(\underline{s}) = \sum_{I \subseteq T} \mathbf{Z}^{(N)}(\underline{s}; I) = \sum_{I \subseteq T} p^{M(\underline{s})} \mathbf{Z}^{(N)}(\underline{s}; I, 0) \mathbf{Z}^{(N)}(\underline{s}; T \setminus I, 1),$$

see Remark 4. Now, by Propositions 1–2 and Lemma 9, for any $I \subseteq T$, $\mathbf{Z}^{(N)}(\underline{s}; I, 0)$ and $\mathbf{Z}^{(N)}(\underline{s}; T \setminus I, 1)$ are holomorphic functions of $\underline{s} \in H_0(\mathbb{C})$, which is an open and connected subset, and consequently the analytic continuations of the integrals $\mathbf{Z}^{(N)}(\underline{s}; I, 0)$ and $\mathbf{Z}^{(N)}(\underline{s}; T \setminus I, 1)$ and formula (3.33) give rise to an analytic continuation of $\mathbf{Z}^{(N)}(\underline{s})$ with the announced properties.

(2) It follows from formula (3.33) by Corollary 2 and Remark 12. \square

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